VI. The Geometry of CR-Manifolds

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Introduction

The geometry of CR-manifolds goes back to Poincaré and received a great attention in the works of É. Cartan, Tanaka, Moser, Chern and others (cf. [44]). In this chapter we consider results connected with the equivalence problem for CR-manifolds in its differential geometric aspect and some applications of this.

§ 1. CR-Manifolds

A real manifold $M^1$ is called an (abstract) Cauchy-Riemann manifold or a CR-manifold if at each point $x \in M$ in the tangent space $T_x M$ there is a distinguished subspace $T^c_x M$ and a complex structure on it, both depending

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1 In this Part, we assume, unless we do not say the contrary, that all objects encountered are smooth of class $C^\infty$. 

G. M. Khenkin (ed.), Several Complex Variables III
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smoothly on $x$. The complex dimension of the subspace $T_x^c(M)$ is called the CR-dimension of $M$ and will be denoted by $\text{CR dim } M$.

The collection of subspaces $T_x^c(M)$ generates a complex tangent distribution $T^c(M)$ or a CR-structure. The complexified distribution $T^e(M) \otimes_\mathbb{C} \mathbb{C}$ decomposes into a sum

$$T^e(M) \otimes_\mathbb{C} \mathbb{C} = T^{(0,1)}(M) + T^{(1,0)}(M).$$

If $J_x : T_x^e(M) \to T_x^e(M)$ is the operator giving the complex structure in $T_x^e(M)$ ($J_x^2 = -\text{id}$) then

$$T_x^{(0,1)}(M) = \{X + iJ_x X : X \in T_x^e(M)\},$$
$$T_x^{(1,0)}(M) = \{X - iJ_x X : X \in T_x^e(M)\}.$$

A function $f$ on an open subset $U \subset M$ is called a CR-function if it satisfies the Cauchy-Riemann equations

$$Xf = 0, \quad X \in \Gamma_U(T^{(0,1)}(M)), \quad (1.1)$$

where $\Gamma_U(T^{(0,1)}(M))$ the set of vector fields on $U$ (sections of the bundle $T^{(0,1)}M$ on the set $U$) and $Xf$ stands for differentiation of $f$ along $X$.

A map between CR-manifolds $f : M_1 \to M_2$ is termed a CR-map if it induces a homomorphism of complex vector bundles $T^e(M_1) \to T^e(M_2)$.

The notion of CR-manifold arose in the study of the tangential Cauchy-Riemann equations on real submanifolds of $\mathbb{C}^n$ (cf. Vol. 7, Chap. II).

Let $M$ be a real submanifold of $\mathbb{C}^n$ with coordinates $(z_1, \ldots, z_N)$. Set

$$T_x^e(M) = T_x(M) \cap JT_x(M), \quad x \in M,$$

where $J$ is the operation of multiplication by the imaginary unit in $\mathbb{C}^N$. On the space $T_x^e(M)$ we have an induced complex structure. Generally speaking, the dimension of $T_x^e(M)$ may depend on $x \in M$; if it is constant then $M$ is a CR-manifold. In this case we say that the CR-structure on $M$ is induced by the complex structure of the ambient space. For manifolds $M \subset \mathbb{C}^N$ with an induced CR-structure, the bundle $T^{(0,1)}(M)$ consists of complex tangent vectors which can be expressed linearly in $\partial/\partial z_1, \ldots, \partial/\partial z_N$, and equation (1.1) reduces to the tangential Cauchy-Riemann equation on $M$.

It follows from this description of $T^{(0,1)}(M)$ for $M \subset \mathbb{C}^N$ that

$$[X, Y] \in \Gamma_U(T^{(0,1)}(M)), \quad \text{provided } X, Y \in \Gamma_U(T^{(0,1)}(M)). \quad (1.2)$$

Condition (1.2) for an abstract CR-manifold will be called the integrability condition for the CR-structure [20]. It is a necessary condition for the existence of an imbedding of $M$ into $\mathbb{C}^N$ such that the CR-structure on $M$ is induced by the complex structure on $\mathbb{C}^N$. In the real-analytic case this condition is also sufficient for the existence of a local imbedding of $M$ into $\mathbb{C}^N$ (cf. [3], [30]). As Nirenberg [27] has demonstrated, for smooth CR-manifolds the last statement is in general not true. From now on we will only consider integrable CR-structures.