Formulation

The classical method of Trefftz consists of minimizing the energy error over the finite-energy solutions of the equilibrium equations. The method of Reissner [4] (also [3]) is equivalent. Analysis of the methods and of their coupling with domain methods requires an alternative formulation of the theory of elasticity. The three-dimensional case is considered (similar results hold in two dimensions). The elastic body occupies a region \( \Omega \) with Lipschitz boundary \( \Gamma \). The notation \( H^S(\Omega) \) denotes the product Sobolev space \( H^S(\Omega)^3 \). The strain-displacement and stress-strain relations are given by:

\[
e_{ij}(u) = (u_{i,j} + u_{j,i}) \quad (1)
\]
\[
\sigma_{ij}(u) = c_{ijkl} e_{kl} \quad (2)
\]

where \( c_{ijkl} \) are the elasticities, \( u \in H^1(\Omega) \) is a finite-energy displacement vector [1], a comma denotes differentiation and a repeated subscript implies summation. The inner-product

\[
(u,v) = \int_{\Omega} \sigma_{ij}(u) e_{ij}(v) \, dx + \int_{\Gamma} \nabla u^T \nabla v \, ds + \int_{\Gamma} (\nabla u)^T v \, ds + \int_{\Gamma} (\nabla v)^T u \, ds \quad (3)
\]

is introduced on the space \( H^1(\Omega) \), where \( T \) denotes transpose and \( x \) denotes position. The set of finite-energy solutions; \( \{ u \in H^1(\Omega) : \sigma_{ij}(u), i,j=0 ; j=1,3 \} \), is the Hilbert space \( F^1(\Omega) \), the orthogonal complement of \( H_0^1(\Omega) \) with respect to the inner-product (3) above [1]. The convergence results are formulated with respect to the Hilbert space:

\[
G^1(\Omega) = \{ u \in F^1(\Omega) : \int_{\Gamma} \nabla u \, ds = 0 , \int_{\Gamma} \nabla u \, ds = 0 \}
\]

(which excludes rigid-body displacements and simplifies analysis) [1]. The spaces:

\[
G^{1/2}(\Gamma) = \{ \xi \in H^{1/2}(\Gamma) : \langle a + b \times \xi , \xi \rangle = 0 ; a,b \in \mathbb{R}^3 \}
\]
\[
G^{-1/2}(\Gamma) = \{ \rho \in H^{-1/2}(\Gamma) : \langle \rho , a + b \times \xi \rangle = 0 ; a,b \in \mathbb{R}^3 \}
\]

are the spaces of traces of displacements in \( G^1(\Omega) \) and of their tractions, respectively, where \( \langle , \rangle \) denotes the duality pairing on \( H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma) \).
which can be identified with the $L^2(\Gamma)^3$ inner-product [1]. Existence/ uniqueness of solutions in $G^1(\Omega)$ of boundary-value problems with data in the boundary spaces follows [1] from bijectivity of the trace and traction operators.

**Convergence of Boundary-Galerkin Methods**

The norm (denoted by $\| \cdot \|$) induced by the inner-product (3) on $G^1(\Omega)$ is the strain-energy. The span of $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ where $\{\mathbf{w}_i\}$ form a basis of $G^1(\Omega)$ is denoted by $S_n$. If $u_\infty \in G^1(\Omega)$ is the solution of the traction problem and if

$$\min_{v_n \in S_n} \| u - v_n \|$$

occurs at $u_\infty$, then $u_\infty$ converges to $u$ (- the Trefftz method is convergent)[1]. Hlavacek [3] proved the convergence of a minimizing sequence in 1971 without establishing its existence. This follows from the above analysis [1]. The Trefftz system of equations has the form:

$$< \mathbf{p} - \mathbf{t}(u_\infty), \mathbf{w}_i > = 0 \quad ; \quad i = 1, 2, \ldots, n$$

(4)

where $\mathbf{p}$ is the specified traction and $\mathbf{t}(u_\infty)$ is the traction corresponding to $u_\infty$, through (1) and (2). The system (4) is the (boundary) Galerkin discretization of the abstract Principle of Virtual Work:

$$< \mathbf{p} - \mathbf{t}(u), v > = 0 \quad ; \quad v \in G^1(\Omega)$$

which, in turn, is the Euler equation corresponding to the boundary variational principle (a special case of Reissner's principle):

$$\min_{v \in G^1(\Omega)} < \mathbf{p} - \mathbf{t}(u), v >$$

which establishes the equivalence and convergence of the three methods [1]. A similar analysis applies to the displacement boundary-value problem [1]. The analysis of boundary-Galerkin methods for mixed problems remains to be carried out. Numerical evidence would suggest that convergence occurs [2].

**Hybrid Finite Element methods**

One possible approach to the coupling of Boundary-element and Finite-element methods is that of a simplified hybrid method. Consider the model problem:

$$\sigma_{ij,j} = f_i \quad \text{in} \quad \Omega,$$

$$u_i = 0 \quad \text{in} \quad \Gamma$$

Let $\Omega_2$ (the 'boundary element') lie in the interior of $\Omega$ and let $\Omega_1 = \Omega \setminus \Omega_2$ (the finite element region). The boundary of $\Omega_2$ is denoted by $\Gamma_0$ and the subspace of $H^1(\Omega_1)$ with zero trace on $\Gamma$ by $\mathbb{E}(\Omega_1)$. It is assumed that supp($\mathbf{f}$) is contained in $\Omega_1$. The following hybrid virtual work principle is based on