Chapter VIII. Spectral Analysis of Singularities

Summary

In Chapter VII we have seen that a distribution $u$ of compact support is smooth if and only if the Fourier transform $\hat{u}$ is rapidly decreasing. If $u$ is not smooth we can use the set of directions where $\hat{u}$ is not rapidly decreasing to describe which are the high frequency components of $u$ causing the singularities. This analysis turns out to have an invariant and local character. For a distribution $u \in \mathcal{D}'(X)$ on a $C^\infty$ manifold $X$ we are therefore led to define a set

$$WF(u) \subset T^*(X) \setminus 0$$

with projection in $X$ equal to sing supp $u$, which is conic with respect to multiplication by positive scalars in the fibers of $T^*(X)$. We call it the wave front set of $u$ by analogy with the classical Huyghens construction of a propagating wave. In this construction one assumes that the location and oriented tangent plane of a wave is known at one instant of time and concludes that at a later time it has been translated in the normal direction with the speed of light. The data are thus precisely rays in the cotangent bundle.

The advantages of the notion of wave front set are manifold. First of all it allows one to extend a number of operations on distributions. For example, the restriction of $u \in \mathcal{D}'(X)$ to a submanifold $Y$ of $X$ can always be defined when the normal bundle of $Y$ does not meet $WF(u)$, that is, high frequency components of $u$ remain of high frequency after restriction to $Y$. Secondly, differential operators and to some extent their fundamental solutions are local even with respect to the wave front set. This leads to important simplifications in their study known as microlocal analysis.

Section 8.1 gives the basic definitions of the wave front set and some important examples. In Section 8.2 we then reconsider the operations defined in Chapters III–VI from our new point of view. Thus we obtain extended definitions of composition and multiplication as well as more precise information on the singularities of the
results of these operations. In Section 8.3 we prove the simplest facts on the wave front set of solutions of linear partial differential equations, in particular that the wave front set is included in the union of the characteristic set and the wave front set of the right-hand side. Note that since the characteristic set $\mathcal{C}^* \mathcal{X} \setminus 0$ usually projects onto $\mathcal{X}$ it is not possible to give a satisfactory statement of this result without the notion of wave front set. When the principal part is real and has constant coefficients we also show that the wave front set is invariant under the bicharacteristic flow, which in the case of the wave equation reduces to the Huyghens construction above and so justifies our terminology.

One can also consider a stricter classification of singularities, such as the set $\text{sing} \supp u$ of points where $u$ is not a real analytic function. This set too admits a spectral decomposition to a set $WF_A(u) \subset T^*(\mathcal{X}) \setminus 0$, which is defined in Section 8.4 and studied in Sections 8.5 and 8.6. In particular this notion allows one to state a more precise form of the uniqueness of analytic continuation: If a distribution vanishes on one side of a $C^1$ hypersurface $Y$ and the normal of $Y$ at $y$ is not in $WF_A(u)$, then $u$ vanishes in a neighborhood of $y$. In other words, the normals of the boundary of $\supp u$ must be in $WF_A(u)$ where the boundary is in $C^1$. In Section 8.5 we also give a notion of normal to the boundary of a general closed set making this statement valid in general. This concept is discussed geometrically at some length in preparation for some later applications. The first comes already in Section 8.6 where various generalizations of the theorem of Holmgren on unique continuation of solutions to partial differential equations with analytic coefficients are given.

In Section 8.7 finally we discuss the analytic wave front set for distributions obtained as limits of $F(x + iy)\gamma$ where $F$ is analytic and $y \to 0$ in a cone such that the zeros of $F$ are only encountered in the limit. The results are useful in the study of the Cauchy problem in Chapter XII.

8.1. The Wave Front Set

If $v \in \mathcal{E}'(\mathbb{R}^n)$ we can decide whether $v$ is in $C^\infty_0$ by examining the behavior of the Fourier transform $\hat{v}$ at $\infty$. In fact, if $v \in C^\infty_0(\mathbb{R}^n)$ then

$$|\hat{v}(\xi)| \leq C_N(1 + |\xi|)^{-N}, \quad N = 1, 2, \ldots, \xi \in \mathbb{R}^n,$$

by Lemma 7.1.3. Conversely, if (8.1.1) is fulfilled then $v \in C^\infty_0$ by Fourier's inversion formula (7.1.4). (See also Theorem 7.3.1.) For a