MARKETS WITH INFINITELY MANY COMMODITIES

We have defined commodities as physical goods which may differ in the location or time at which they are produced or consumed, or in the state of the world in which they become available. If we allow an infinite variation in any of these contingencies, then we are naturally led to consider economies with infinitely many commodities.

T. F. Bewley’s 1972 paper [16] is the seminal article on the existence of Walrasian equilibria in economies with a finite number of agents and infinitely many commodities. Equally important for our research is a little noticed 1970 paper by B. Peleg and M. E. Yaari [53] on the existence of competitive equilibria in an exchange economy with a countable number of commodities. A comparison of these two disparate approaches to the existence problem, in economies with infinite dimensional commodity spaces, is an excellent introduction to the merits of a Riesz space analysis of general equilibrium models.

The Peleg–Yaari model is a model of an intertemporal infinite horizon economy in discrete time—where the state of the world is known; agents are assumed to have perfect foresight regarding prices; there are a finite number of agents and a countable number of time periods; and in each period a single perishable good is available for consumption. Distinguishing between consumption today and consumption tomorrow, gives rise to a countable number of commodities. Hence, the commodity space is $\mathbb{R}^\infty$ and each agent’s consumption set is $\mathbb{R}^+_\infty$, where she has a preference relation and an initial endowment. The intended interpretation of this model is a decentralized model of economic growth and agents are thought to be national economies.

Agents are assumed to be impatient in the sense of I. Fisher [28], i.e., they prefer present consumption to future consumption. This behavioral assumption on tastes is captured by requiring each agent’s preference to be continuous in the product topology. In addition, preferences are assumed to be strictly convex and strictly monotone. Prices in this model should correspond to interest rates between periods; hence, in the Peleg–Yaari model, prices are defined as nonnegative sequences of real numbers that give finite valuation to the social endowment. It follows from the assumption of strict monotonicity of preferences that equilibrium interest rates are positive in each period—and thus are never in the dual space of $\mathbb{R}^\infty$. 

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Assuming perfect foresight, the notion of competitive equilibrium in the Peleg-Yaari model is the same as that in the Arrow-Debreu model of Chapter 1. The Peleg-Yaari proof follows that of H. E. Scarf—they first show that the core is non-empty; then they prove that Edgeworth equilibria exist; and finally they demonstrate the existence of prices that support an Edgeworth equilibrium as a Walrasian equilibrium. The last part of their argument is much more delicate than the similar step in the Debreu-Scarf paper [24], since $\mathcal{R}_+^*$ has empty interior in the product topology which prevents the straightforward application of the separating hyperplane theorem. The fact that the economically interesting topologies on a commodity space $E$ typically give rise to an empty interior for $E^+$ is the essential difference between the standard Arrow-Debreu model and the general equilibrium models which are the principal concern of this monograph.

In contrast to the Peleg-Yaari model, T. F. Bewley formulates his model in terms of a dual pair of locally convex spaces $(E, E')$ which correspond, respectively, to the commodity and price spaces. For the economic situation considered by Peleg and Yaari, Bewley’s model specializes to the dual pair $(\ell_\infty, \mathcal{B}_1)$. Agents’ consumption sets are $\ell_1^+$ and preferences are strictly convex and strictly monotone. Bewley also assumes that agents are impatient by requiring preferences to be lower semicontinuous with respect to the Mackey topology for the dual pairing $(\ell_\infty, \ell_1)$.

The Hewitt-Yosida representation theorem states that every linear functional in $\mathcal{B}_1$ can be expressed as the sum of a linear functional in $\ell_1$ and a purely finitely additive linear functional. Purely finitely additive functionals cannot be interpreted as defining interest rates between periods. Hence, Bewley’s proof of existence is in two parts. First, he demonstrates the existence of a Walrasian equilibrium with supporting prices in $\mathcal{B}_1$. Then, using the impatience assumption he shows that the $\ell_1$ part of the supporting prices is nontrivial and supports the given allocation as a competitive allocation. To prove the existence of a Walrasian equilibrium with prices in $\mathcal{B}_1$, Bewley restricts agents’ characteristics to the finite dimensional subspaces of $\ell_\infty$ that contain the initial and total endowments. For each of the standard Arrow-Debreu exchange economies, there is a Walrasian equilibrium by the Arrow–Debreu existence theorem. Finally, he extracts a convergent subnet from this net of allocations and prices. The limit is the desired equilibrium allocation and price.

To compare the two models, we first observe that $\ell_\infty$ is a linear subspace of $\mathcal{R}_\infty$. Invoking the Riesz space structure of $\mathcal{R}_\infty$, much more is true. That is, $\ell_\infty$ is a principal ideal of $\mathcal{R}_\infty$. For our purposes, a more interesting principal ideal of $\mathcal{R}_\infty$ is $A_\omega$, where $\omega$ is the total endowment of the agents in the Peleg-Yaari model. Restricting the preferences of agents in the Peleg-Yaari model to $A_\omega$ and noting that $A_\omega$ and $\ell_\infty$ are both AM-spaces, we might expect (by Bewley’s theorem) that this restricted economy has a Walrasian equilibrium with respect to the duality $(A_\omega, A_\omega')$. This conjecture is true, but our proof does not follow Bewley’s limiting argument. Instead, we shall use Scarf’s argument for demonstrating the existence of Walrasian equilibria in the Arrow-Debreu model.

First, the order interval $[0, x]$ of $\mathcal{R}_\infty$ is weakly compact for each $x \in \mathcal{R}_\infty^+$ and—as first observed by Peleg and Yaari—this is sufficient to prove the existence of core allocations. Each agent’s consumption in a core allocation lies in $A_\omega$; and