Chapter I. Fractional Linear Transformations

In this chapter we review the basic properties of fractional linear transformations. For the convenience of the reader, we start from scratch and derive the properties we need. The point of view here is strictly one complex dimensional; isometries of hyperbolic spaces will be developed in Chapter IV.

I.A. Basic Concepts

A.1. We start with some notation. The extended complex plane \( \mathbb{C} \cup \{ \infty \} \) is denoted by \( \hat{\mathbb{C}} \). Every orientation preserving conformal homeomorphism of \( \hat{\mathbb{C}} \) is a fractional linear, or Möbius, transformation; i.e., a transformation of the form

\[
g(z) = \frac{az + b}{cz + d},
\]

where \( a, b, c, d \) are complex numbers and the determinant \( ad - bc \neq 0 \). We denote the group of all fractional linear transformations by \( \mathbb{M} \). Every \( g \in \mathbb{M} \) is either the identity, or has at most two fixed points.

One usually thinks of the transformation \( g(z) \) as the matrix

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}.
\]

This transformation is unchanged if we multiply all four coefficients by the same number \( t \neq 0 \), while of course the matrix is changed. From here on we regard the matrices

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
ta & tb \\
tc & td
\end{pmatrix}
\]

as being the same. More precisely, there is an isomorphism between \( \mathbb{M} \) and \( PGL(2, \mathbb{C}) \), the projectivized group of non-singular \( 2 \times 2 \) matrices with complex entries; an easy calculation shows that composition of maps corresponds to multiplication of matrices. We always regard \( 2 \times 2 \) matrices as being in \( PGL(2, \mathbb{C}) \).

By making judicious use of our projectivizing factor \( t \), we can ensure that the determinant \( ad - bc = 1 \). In this case our identification establishes an iso-
morphism between $\mathbb{M}$ and $\text{PSL}(2, \mathbb{C})$, the group of matrices as above, but with determinant 1.

Throughout this book, unless specified otherwise, we will always write elements of $\mathbb{M}$ as matrices with determinant 1. Of course, there are two ways to do this; that is, as elements of $\mathbb{M}$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$$

are equal.

A.2. We are primarily interested in conjugacy classes of subgroups of $\mathbb{M}$. In $\text{SL}(2, \mathbb{C})$ there is essentially only one conjugation invariant function, the trace, $\text{tr}(\cdot)$. The trace is not well defined in $\mathbb{M}$, but its square is; we write $\text{tr}^2(g)$ for the square of the trace of $g$; i.e., $\text{tr}^2(g) = (a + d)^2$.

A.3. Every orientation reversing conformal homeomorphism of $\mathbb{C}$ is of the form $g(z) = (az + b)/(cz + d)$, $ad - bc \neq 0$. As in the orientation preserving case, the coefficients are homogeneous, so we can always assume that the determinant, $ad - bc = 1$. There is no standard terminology in the literature for these transformations; we will call them fractional reflections (we reserve the word reflection for a fractional reflection with a circle of fixed points). We denote the group of all fractional linear transformations and fractional reflections by $\mathfrak{M}$. We leave it to the reader to work out the rules for translating composition in $\mathfrak{M}$ into multiplication of matrices.

A.4. We sometimes call the identity element of a group the trivial element. Every non-trivial element of $\mathbb{M}$ has either one or two fixed points; fractional reflections are somewhat more complicated.

We regard lines in $\mathbb{C}$ as being circles in $\mathfrak{C}$ which pass through $\infty$.

**Proposition.** The fixed point set of a fractional reflection is either empty, one point, two points, or a circle in $\mathfrak{C}$.

**Proof.** We write the fractional reflection as

$$g(z) = (az + b)/(cz + d),$$

and we first assume that $g(\infty) = \infty$, or equivalently, that $c = 0$. Using homogeneity, we can also assume that $d = 1$. We now have to solve the equation $z = az + b$. Separating this equation into its real and imaginary parts, we get two inhomogeneous linear equations in two unknowns. The solution set is either empty, one point, or a line. Combining these with the known fixed point at $\infty$, we obtain a fixed point set that is either one point, two points, or a circle.

We next take up the case that $c \neq 0$; assume that $c = 1$. We now need to solve the equation $|z|^2 + dz - a\overline{z} - b = 0$. Setting the real part equal to zero, we get