Chapter VI. Geometrically Finite Groups

This chapter is an exploration of geometrically finite discrete subgroups of \( \mathbb{M} \); that is, groups that have (convex) fundamental polyhedra in \( \mathbb{H}^3 \) with finitely many sides. One of our main objectives is to give a criterion for a group to be geometrically finite in terms of its action at the limit set; this criterion will then be used in Chapter VII to show that, under suitable conditions, the combination of two geometrically finite groups is again geometrically finite.

A geometrically finite group can also be described in terms of the underlying manifold \( (\mathbb{H}^3 \cup \Omega(G))/G \); this manifold need not be compact, but its non-compact ends can be completely described in terms of a finite set of topologically distinct possibilities.

VI.A. The Boundary at Infinity of a Fundamental Polyhedron

A.1. Let \( G \) be a discrete subgroup of \( \mathbb{L}^n \), and let \( D \subset \mathbb{H}^n \) be a fundamental polyhedron for \( G \). By definition, \( D \) is convex, so the different sides of \( D \) lie on different hyperplanes. In fact, different sides are identified by different elements of \( G \).

**Proposition.** Let \( s_1 \neq s_2 \) be sides of \( D \), and let \( g_1 \) and \( g_2 \) be the corresponding side pairing transformations; i.e., there are sides \( s_1' \) and \( s_2' \), so that \( g_m(s_m) = s'_m \). Then \( g_1 \neq g_2 \).

**Proof.** The hyperplane on which \( s_1' \) lies separates \( D \) from \( g_1(D) \); in particular, it separates \( s_2' \) from \( g_1(s_2) \). Hence \( g_1(s_2) \) is not a side of \( D \).

A.2. For a polyhedron \( D \subset \mathbb{B}^n \) we denote the relative boundary of \( D \) in \( \mathbb{B}^n \) by \( \partial D \); we denote the intersection of the Euclidean boundary of \( D \) with \( \partial \mathbb{B}^n \) by \( \partial^D \). The relative interior of \( \partial D \) in \( \partial \mathbb{H}^n \) is denoted by \( \mathring{\partial} D \).

If \( x \in \partial D \), and \( x \) lies on the boundary of the side \( s \) of \( D \), then we say that \( s \) abuts \( x \).

A.3. Thus far, the term "fundamental domain" is defined only for subgroups of \( \mathbb{M} \) (see II.G). The generalization to subgroups of \( \mathbb{L}^n \), acting on \( \mathbb{E}^{n-1} \), is obvious.
**Proposition.** If $D$ is a fundamental polyhedron for the discrete subgroup $G$ of $\mathbb{P}^n$, then $^c\mathcal{D}D$ is a fundamental domain for $G$.

**Proof.** It is immediate from the definition that no two points of $^c\mathcal{D}D$ are equivalent under $G$. The sides of $^c\mathcal{D}D$ are the boundaries of the sides of $D$. These are paired by the same elements that pair the sides of $D$.

In $\mathbb{B}^n$, only finitely many supporting hyperplanes of sides of $D$ meet any compact set. If $\{s_m\}$ is a sequence of sides of $D$, and $Q_m$ is the supporting hyperplane of $s_m$, then $\text{dia}_E(Q_m) \to 0$; hence $\text{dia}_E(\partial s_m) \to 0$.

If $x$ is a point on the boundary of $^c\mathcal{D}D$, where $x$ does not lie on any side, then there is a sequence of sides of $D$ accumulating to $x$; hence there is a sequence of side pairing transformations $\{g_m\}$ so that for all $z \in ^c\mathcal{D}D$, $g_m(z) \to x$. Hence $x$ is a limit point of $G$.

Let $x$ be a point on the boundary of $^c\mathcal{D}D$, where $x \in \Omega$. Then $x$ lies on a side of $^c\mathcal{D}D$, so there is a side $s$ of $D$ abutting $x$. There are only finitely many translates of $D$ in a neighborhood of $s$, hence there are only finitely many translates of $^c\mathcal{D}D$ in a neighborhood of $x$.

Finally, if $z \in \Omega(G)$, then choose a sequence of points $\{x_m\}$ in $\mathbb{B}^n$, with $x_m \to z$. For each $x_m$ there is an element $g_m \in G$, with $x_m \in g_m(D)$. There are only a finite number of distinct elements in this sequence, for otherwise, we would have $\text{dia}_E g_m(D) \to 0$, contradicting the assumption that $z \in \Omega$. Hence there is a $g \in G$ with $g(z)$ in the closure of $^c\mathcal{D}D$. \hfill \square

**A.4.** In $\mathbb{B}^n$, a horosphere $S$ is a Euclidean $(n - 1)$-sphere which is tangent to $\partial \mathbb{B}^n$, and which, except for the point of tangency, lies in $\mathbb{B}^n$. The open Euclidean ball bounded by $S$ is called a horoball.

The point of tangency of $S$ with $\partial \mathbb{B}^n$ is the center, or vertex, of $S$. It is also the center, or vertex, of the horoball bounded by $S$.

A horosphere in $\mathbb{H}^n$, centered at a finite point $x$, is a Euclidean sphere tangent to $\partial \mathbb{H}^n$, which, except for the point of tangency, lies in $\mathbb{H}^n$. A horosphere centered at $\infty$ is an Euclidean plane parallel to $\partial \mathbb{H}^n$.

**A.5. Proposition.** Let $G$ be a discrete subgroup of $\mathbb{M}$, where $G$ contains $j(z) = z + 1$. Then the horoball $T = \{(z,t) \in \mathbb{H}^3 | t > 1\}$ is precisely invariant under $\text{Stab}(\infty)$.

**Proof.** Let $J = \text{Stab}(\infty)$; by II.C.6, no element of $J$ is loxodromic. Hence every element of $J$ lies in $\mathbb{A}^2$, and every element of $\mathbb{A}^2$ keeps every horosphere centered at $\infty$ invariant.

If $$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$ is any element of $G$, then by II.C.5, either $c = 0$, in which case $g \in J$, or $|c| \geq 1$. In the latter case, the radius of the isometric circle of $g$ is at most one. Write