Chapter VI. Geometrically Finite Groups

This chapter is an exploration of geometrically finite discrete subgroups of $\mathbb{M}$; that is, groups that have (convex) fundamental polyhedra in $\mathbb{H}^3$ with finitely many sides. One of our main objectives is to give a criterion for a group to be geometrically finite in terms of its action at the limit set; this criterion will then be used in Chapter VII to show that, under suitable conditions, the combination of two geometrically finite groups is again geometrically finite.

A geometrically finite group can also be described in terms of the underlying manifold $(\mathbb{H}^3 \cup \Omega(G))/G$; this manifold need not be compact, but its non-compact ends can be completely described in terms of a finite set of topologically distinct possibilities.

VI.A. The Boundary at Infinity of a Fundamental Polyhedron

A.1. Let $G$ be a discrete subgroup of $\mathbb{L}^n$, and let $D \subset \mathbb{H}^n$ be a fundamental polyhedron for $G$. By definition, $D$ is convex, so the different sides of $D$ lie on different hyperplanes. In fact, different sides are identified by different elements of $G$.

Proposition. Let $s_1 \neq s_2$ be sides of $D$, and let $g_1$ and $g_2$ be the corresponding side pairing transformations; i.e., there are sides $s_1'$ and $s_2'$, so that $g_m(s_m) = s_m'$. Then $g_1 \neq g_2$.

Proof. The hyperplane on which $s_1'$ lies separates $D$ from $g_1(D)$; in particular, it separates $s_2'$ from $g_1(s_2)$. Hence $g_1(s_2)$ is not a side of $D$.

A.2. For a polyhedron $D \subset \mathbb{B}^n$ we denote the relative boundary of $D$ in $\mathbb{B}^n$ by $\partial D$; we denote the intersection of the Euclidean boundary of $D$ with $\partial \mathbb{B}^n$ by $\partial ^* D$. The relative interior of $\partial D$ in $\partial \mathbb{H}^n$ is denoted by $\partial^* D$.

If $x \in \partial D$, and $x$ lies on the boundary of the side $s$ of $D$, then we say that $s$ abuts $x$.

A.3. Thus far, the term "fundamental domain" is defined only for subgroups of $\mathbb{M}$ (see II.G). The generalization to subgroups of $\mathbb{L}^n$, acting on $\mathbb{E}^{n-1}$, is obvious.
Proposition. If \( D \) is a fundamental polyhedron for the discrete subgroup \( G \) of \( \mathbb{P}^n \), then \( \partial D \) is a fundamental domain for \( G \).

Proof. It is immediate from the definition that no two points of \( \partial D \) are equivalent under \( G \). The sides of \( \partial D \) are the boundaries of the sides of \( D \). These are paired by the same elements that pair the sides of \( D \).

In \( \mathbb{B}^n \), only finitely many supporting hyperplanes of sides of \( D \) meet any compact set. If \( \{s_m\} \) is a sequence of sides of \( D \), and \( Q_m \) is the supporting hyperplane of \( s_m \), then \( \text{diam}_E(Q_m) \to 0 \); hence \( \text{diam}_E(\partial s_m) \to 0 \).

If \( x \) is a point on the boundary of \( \partial D \), where \( x \) does not lie on any side, then there is a sequence of sides of \( D \) accumulating to \( x \); hence there is a sequence of side pairing transformations \( \{g_m\} \) so that for all \( z \in \partial D \), \( g_m(z) \to x \). Hence \( x \) is a limit point of \( G \).

Let \( x \) be a point on the boundary of \( \partial D \), where \( x \in \mathbb{O} \). Then \( x \) lies on a side of \( \partial D \), so there is a side \( s \) of \( D \) abutting \( x \). There are only finitely many translates of \( D \) in a neighborhood of \( s \), hence there are only finitely many translates of \( \partial D \) in a neighborhood of \( x \).

Finally, if \( z \in \mathbb{O}(G) \), then choose a sequence of points \( \{x_m\} \) in \( \mathbb{B}^n \), with \( x_m \to z \). For each \( x_m \) there is an element \( g_m \in G \), with \( x_m \in g_m(D) \). There are only a finite number of distinct elements in this sequence, for otherwise, we would have \( \text{diam}_E g_m(D) \to 0 \), contradicting the assumption that \( z \in \mathbb{O} \). Hence there is a \( g \in G \) with \( g(z) \) in the closure of \( \partial D \). \( \square \)

A.4. In \( \mathbb{B}^n \), a horosphere \( S \) is a Euclidean \((n - 1)\)-sphere which is tangent to \( \partial \mathbb{B}^n \), and which, except for the point of tangency, lies in \( \mathbb{B}^n \). The open Euclidean ball bounded by \( S \) is called a horoball.

The point of tangency of \( S \) with \( \partial \mathbb{B}^n \) is the center, or vertex, of \( S \). It is also the center, or vertex, of the horoball bounded by \( S \).

A horosphere in \( \mathbb{H}^n \), centered at a finite point \( x \), is a Euclidean sphere tangent to \( \partial \mathbb{H}^n \), which, except for the point of tangency, lies in \( \mathbb{H}^n \). A horosphere centered at \( \infty \) is a Euclidean plane parallel to \( \partial \mathbb{H}^n \).

A.5. Proposition. Let \( G \) be a discrete subgroup of \( \mathbb{M} \), where \( G \) contains \( j(z) = z + 1 \). Then the horoball \( T = \{(z, t) \in \mathbb{H}^3 \mid t > 1\} \) is precisely invariant under \( \text{Stab}(\infty) \).

Proof. Let \( J = \text{Stab}(\infty) \); by II.C.6, no element of \( J \) is loxodromic. Hence every element of \( J \) lies in \( \mathbb{A}^2 \), and every element of \( \mathbb{A}^2 \) keeps every horosphere centered at \( \infty \) invariant.

If
\[
g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]
is any element of \( G \), then by II.C.5, either \( c = 0 \), in which case \( g \in J \), or \( |c| \geq 1 \). In the latter case, the radius of the isometric circle of \( g \) is at most one. Write