Hydrodynamic Stability and Turbulence (1922–1948)

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Part I

Heisenberg’s long paper (No. 2, p. 31 *), devoted in part to a study of the stability of plane-parallel laminar flows, is, by any standards, an important and fundamental contribution to the subject. In this paper, besides describing the essential steps that must be taken to solve the underlying mathematical problem, Heisenberg devises for the first time the use of “inner” and “outer” approximations with suitable matching conditions for the solution of ordinary differential equations (of order two or higher) with a turning point — a method later to be described by the initials “W.K.B.”. And beneath the mathematical developments, one can discern the operation of a powerful physical insight. Heisenberg himself described later the physical and the mathematical ideas in this early work of his in an address to the International Congress of Mathematicians in 1950 [1]. Perhaps a somewhat more technical account of these ideas may be useful as an introduction to his paper.

The problem, Heisenberg considers, is the classical one of the stability of plane laminar flow (in the $x$-direction, say) confined between two parallel planes (which we may assume to be at $y = \pm 1$, by measuring distances in units of the separation, $d$, between the two planes) with a velocity profile $w(y)$ (measured in units of the velocity $U$ at, say, $y = 0$). Restricting oneself to two-dimensional disturbances (which by Squire’s theorem entails no loss of generality) and expressing the perturbations, $u$ and $v$, in the velocities in the directions $x$ and $y$ in terms of a stream function, $\psi(x,y)$, in the manner

$$ u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}, \quad (1) $$

and putting

$$ \psi(x,y) = \varphi(y) \exp[i\alpha(x-ct)], \quad (2) $$

one obtains the Orr-Sommerfeld equation,

$$ (w - c)(\varphi'' - \alpha^2 \varphi) - w'' \varphi = -\frac{i}{\alpha R} (\varphi''' - 2 \alpha^2 \varphi'' + \alpha^4 \varphi), \quad (3) $$

where primes denote differentiations with respect to $y$ and

* All page numbers refer to pages in this volume. (Editors)
\[ R = \frac{U d}{v} \]  \hspace{1cm} \text{(4)}

is the Reynolds number of the flow \((v, \text{denoting the kinematic viscosity}).

In the inviscid limit of zero viscosity and infinite Reynolds number, equation (3) reduces to Rayleigh's equation,

\[ (w - c)(\phi'' - \alpha^2 \phi) - w'' \phi = 0. \]  \hspace{1cm} \text{(5)}

In the context of this equation, Rayleigh had shown that a necessary and sufficient condition for instability is that \(w(y)\) has a point of inflexion in the interval, \(-1 < y < +1\). For Poiseuille flow,

\[ w(y) = 1 - y^2, \]  \hspace{1cm} \text{(6)}

there is no point of inflexion, and the inviscid flow is stable. But the flow is unstable when viscosity is present and the Reynolds number exceeds a certain critical value, \(R_c\). The problem is to ascertain this critical Reynolds number for the onset of instability, and Heisenberg proceeds as follows.

First, Heisenberg seeks a solution of the Orr-Sommerfeld equation by making the substitutions,

\[ \phi = \exp \left\{ \int g(y) \, dy \right\} \]  \hspace{1cm} \text{(7)}

and

\[ g(y) = (\alpha R)^{1/2} g_0(y) + g_1(y) + O \left[ (\alpha R)^{-1/2} \right]. \]  \hspace{1cm} \text{(8)}

These substitutions provide approximations to two solutions, \(\phi_1\) and \(\phi_2\), which are rapidly varying when \(\alpha R\) is large. (The method of approximation sought here is the same as the Wentzel-Kramers-Brillouin approximation familiar in the quantum theory.) To obtain approximations to two additional solutions, \(\phi_3\) and \(\phi_4\), Heisenberg considers an expansion in powers of \((\alpha R)^{-1}\) so that in the lowest order, these approximations are the solutions of Rayleigh's equation (5). These "inviscid" approximations are further expanded in powers of \(\alpha^2\). This is the weakest point in Heisenberg's analysis. In the more recent investigations, one uses instead the exact solutions obtained by numerical methods.

Heisenberg next considers the manner of connecting the approximate solutions obtained on either side of the turning point \(y_0\) where \(w(y_0) - c = 0\). He accomplishes this matching by introducing a "stretched variable",

\[ \eta = (y - y_0)(\alpha R)^{1/3}, \]  \hspace{1cm} \text{(9)}

and expanding \(\phi_1\) and \(\phi_2\) in powers of \((\alpha R)^{-1/3}\). This procedure leads to what one now calls the "inner" expressions of the solutions. In dealing with \(\phi_1\) and \(\phi_2\) only the first approximation is required; and Heisenberg obtains this approximation from Hopf's solution [2] for linear profiles. When \(\alpha \ll 1\), \(\phi_3 = w - c\) and no connecting relation is needed. But in the case of \(\phi_4\), two terms of the inner expansion are required; and they must be matched on either side of \(y_0\) to

\[ \phi_4 = (w - c) \int_{-1}^{y} dy (w - c)^{-2}. \]  \hspace{1cm} \text{(10)}