Polynomials and rational fractions

Throughout this chapter \( A \) denotes a commutative ring.

§ 1. POLYNOMIALS

1. Definition of polynomials

Let \( I \) be a set. We recall (III, p. 452) that the free commutative algebra on \( I \) over \( A \) is denoted by \( A[(X_i)_{i \in I}] \) or \( A[X_i]_{i \in I} \). The elements of this algebra are called \textit{polynomials} with respect to the indeterminates \( X_i \) (or in the indeterminates \( X_j \) with coefficients in \( A \). Let us recall that the indeterminate \( X_i \) is the canonical image of \( i \) in the free commutative algebra on \( I \) over \( A \); sometimes it is convenient to denote this image by another symbol such as \( X_i', Y_i, T_i \), etc. This convention is often introduced by a phrase such as : « Let \( Y = (Y_i)_{i \in I} \) be a family of indeterminates » ; in this case the algebra of polynomials in question is denoted by \( A[Y] \). When \( I = \{1, 2, \ldots, n\} \), one writes \( A[X_1, X_2, \ldots, X_n] \) in place of \( A[(X_i)_{i \in I}] \).

For \( v \in N(I) \) we put

\[
X^v = \prod_{i \in I} X_i^{v_i}.
\]

Then \( (X^v)_{v \in N(I)} \) is a basis of the \( A \)-module \( A[(X_i)_{i \in I}] \). The \( X^v \) are called \textit{monomials} in the indeterminates \( X_i \). For \( v = 0 \) we obtain the unit element of \( A[(X_i)_{i \in I}] \). Every polynomial \( u \in A[(X_i)_{i \in I}] \) can be written in exactly one way in the form

\[
u = \sum_{v \in N(I)} \alpha_v X^v
\]

where \( \alpha_v \in A \) and the \( \alpha_v \) are zero except for a finite number ; the \( \alpha_v \) are called the \textit{coefficients} of \( u \); the \( \alpha_v X^v \) are called the \textit{terms} of \( u \) (often the element \( \alpha_v X^v \) is called the term in \( X^v \)), in particular the term of \( \alpha_0 X^0 \), identified with \( \alpha_0 \), is called the \textit{constant term} of \( u \). When \( \alpha_v = 0 \), we say by abuse of language that \( u \) \textit{contains no element} in \( X^v \); in particular when \( \alpha_0 = 0 \), we say that \( u \) is a polynomial \textit{without constant term} (III, p. 453). Any scalar multiple of 1 is called a \textit{constant polynomial}. 

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Let $B$ be a commutative ring and $\rho : A \to B$ a ring homomorphism. We consider $B[(X_i)_{i \in I}]$ as an $A$-algebra by means of $\rho$. Thus the mapping $\sigma$ of $A[(X_i)_{i \in I}]$ into $B[(X_i)_{i \in I}]$ which transforms $\sum \alpha vX^v$ into $\sum \rho(\alpha)vX^v$ is a homomorphism of $A$-algebras; if $u \in A[(X_i)_{i \in I}]$, we sometimes denote by $\rho u$ the image of $u$ by this homomorphism. The homomorphism of $B \otimes_A A[(X_i)_{i \in I}]$ into $B[(X_i)_{i \in I}]$ canonically defined by $\sigma$ transforms, for every $i \in I$, $1 \otimes X_i$ into $X_i$; this is an isomorphism of $B$-algebras (III, p. 449).

Let $M$ be a free $A$-module with basis $(e_i)_{i \in I}$. There exists precisely one unital homomorphism $\varphi$ of the symmetric algebra $S(M)$ into the algebra $A[(X_i)_{i \in I}]$ such that $\varphi(e_i) = X_i$ for each $i \in I$, and this homomorphism is an isomorphism (III, p. 506). This isomorphism is said to be canonical. It allows us to apply to polynomial algebras certain properties of symmetric algebras. For example, let $(I_\lambda)_{\lambda \in L}$ be a partition of $I$. Let $\varphi_\lambda$ be the homomorphism of $P_\lambda = A[(X_i)_{i \in I_\lambda}]$ into $P = A[(X_i)_{i \in I}]$ which transforms $X_i$ (qua element of $P_\lambda$) into $X_i$ (qua element of $P$). Then the $\varphi_\lambda$ define a homomorphism of the algebra $\bigotimes \lambda \in L P_\lambda$ into the algebra $P$, and this homomorphism is an isomorphism (III, p. 503, Prop. 9).

Let $E$ be an $A$-module, and put $E \otimes_A A[(X_i)_{i \in I}] = E[(X_i)_{i \in I}]$. The elements of the $A$-module $E[(X_i)_{i \in I}]$ are called polynomials in the indeterminates $X_i$ with coefficients in $E$. Such a polynomial can be written in just one way as $\sum e_v \otimes X^v$, where $e_v \in E$ and the $e_v$ are zero for all but a finite number of suffixes; we frequently write $e_vX^v$ instead of $e_v \otimes X^v$.

2. Degrees

Let $P = A[(X_i)_{i \in I}]$ be a polynomial algebra. For each integer $n \in \mathbb{N}$ let $P_n$ be the submodule of $P$ generated by the monomials $X^v$ such that $|v| = \sum_{i \in I} v_i$ equals $n$. Then $(P_n)_{n \in \mathbb{N}}$ is a graduation which turns $A[(X_i)_{i \in I}]$ into a graded algebra of type $\mathbb{N}$ (III, p. 459). The homogeneous elements of degree $n$ in $A[(X_i)_{i \in I}]$ are sometimes called forms of degree $n$ with respect to the indeterminates $X_i$.

When we are dealing with the degree of inhomogeneous polynomials, we shall agree to adjoin to the set $\mathbb{N}$ of natural numbers, an element written $-\infty$ and to extend the order relation and the addition of $\mathbb{N}$ to $\mathbb{N} \cup \{-\infty\}$ by the following conventions, where $n \in \mathbb{N},$

\[
-\infty < n , \quad (-\infty) + n = n + (-\infty) = -\infty , \quad (-\infty) + (-\infty) = -\infty .
\]

Let $u = \sum_{v \in \mathbb{N}^{(I)}} \alpha_vX^v$ be a polynomial. The homogeneous component $u_n$ of degree