4 The Mandelbrot Set

For polynomials of second order, \( p(x) = a_2x^2 + a_1x + a_0 \), an almost complete classification of the corresponding Julia sets can be given in terms of the Mandelbrot set. First note that \( p(x) \) is conjugate to \( p_c(z) = z^2 + c \) by means of the coordinate transformation \( x \leftrightarrow z = a_2x + a_1/2 \), with \( c = a_0a_2 + \frac{a_1}{2} \left(1 - \frac{a_1}{2}\right) \).

This transformation shifts the finite critical point \( x = -a_1/2a_2 \) into the origin. It is thus sufficient to study the nature of the Julia sets of \( p_c(z) \).

The point \( \infty \) is a superattractive fixed point of the mapping \( z \mapsto p_c(z) \). The Julia set \( J_c \) for given \( c \in \mathbb{C} \), can therefore be characterized as \( J_c = \partial A(\infty) \). From the theory of Julia and Fatou it follows that \( J_c \) is either connected or a Cantor set \([1]\). This distinction is reflected in the definition of the Mandelbrot set:

\[
M = \{ c \in \mathbb{C} : J_c \text{ is connected} \}.
\]

Figures 3, 4, 6–10, 12 and 14 are examples of connected Julia sets whereas Figs. 11, 13 and 15 show Julia sets with Cantor set structure. Among the connected Julia sets there are those which enclose an interior and others, like Fig. 12, which are dendrites without an inner region.

To compute \( M \), B. B. Mandelbrot employed the powerful results of Julia and Fatou according to which the main dynamical features of a rational mapping can be inferred from the forward orbits of its critical points (see special Section 3): Any attractive or rationally indifferent cycle has in its domain of attraction at least one critical point. But \( p_c(z) \) has only two critical points, \( z = 0 \) and \( \infty \), which are independent of \( c \). The point \( \infty \) is already an attractive fixed point, so only \( 0 \) remains as an interesting critical point to study. By choosing \( c = 1 \), e. g., we see that there are values of \( c \) for which \( 0 \in A(\infty) \), since \( 0 \mapsto 1 \mapsto 2 \mapsto 5 \mapsto 26 \mapsto 677 \mapsto \ldots \). In these cases there cannot be another attractor besides \( \infty \). On the other hand, as the case \( c = 0 \) shows, there are also \( c \) such that there is another attractor: under \( p_0(z) = z^2 \), the point \( z = 0 \) attracts all \( z \) with \( |z| < 1 \), i. e. \( J_0 = S^1 \).

Now according to Julia and Fatou, \( J_c \) is connected if and only if \( 0 \in A(\infty) \), see \([1]\), i. e.

\[
M = \{ c \in \mathbb{C} : p_c^k(0) \not\to \infty \text{ as } k \to \infty \}.
\]

This characterization is very suitable for numerical studies. One chooses a lattice of points \( c \in \mathbb{C} \) and tests for every such \( c \) whether after \( N \) iterations the modulus of the sequence \( 0 \mapsto c \mapsto c^2 + c \mapsto \ldots \) is still below a given bound \( m \).

(For Fig. 2 we took \( N = 1000 \) and \( m = 100 \).)

A. Douady and J. H. Hubbard \([1, 2]\) have found a deep analytic characterization of \( M \). They studied the nature of filled-in Julia sets \( K_c \)

\[
K_c = \{ z \in \mathbb{C} : p_c^k(z) \not\to \infty \text{ as } k \to \infty \},
\]
and noticed that for \( c \in M \) their complements can be mapped onto the complement of the closed unit disk \( \overline{D} \), by means of a conformal mapping \( \varphi_c \).

\[
\varphi_c : \mathbb{C} \setminus K_c \to \mathbb{C} \setminus \overline{D}.
\]

Remarkably, this mapping can be chosen in such a way that

\[
\varphi_c \circ p_c \circ \varphi_c^{-1} = p_0.
\]

Note that locally \( \varphi_c \) is guaranteed by Boettcher's result (2.34). This identifies \( M \) as

\[
M = \{ c \in \mathbb{C} : p_c \text{ on } A(\infty) \text{ is equivalent to } z \mapsto z^2 \}.
\]

The conjugation (4.5) is even possible for \( c \in M \), but then it does not hold in all of \( A(\infty) \). Nevertheless, it can be extended far enough to hold at the point \( z = c \), and by setting

\[
\psi(c) := \varphi_c(c),
\]

we have a mapping \( \psi : \mathbb{C} \setminus M \to \mathbb{C} \setminus \overline{D} \) which is a conformal isomorphism. In this way Douady and Hubbard demonstrated that

\[
M \text{ is a connected set}
\]

(i.e. \( M \) is not contained in the union of two disjoint open nonempty sets). It is still unknown, however, whether \( M \) is also locally connected, i.e. whether any piece \( U \cap M \) of \( M \) (\( U \subset \mathbb{C} \) open) has the property that for any \( z \in U \cap M \) there is a neighborhood \( V \subset U \), \( z \in V \), such that \( V \setminus M \) is connected. The difficulty is that one cannot draw on properties of \( K_c \) because there are \( c \) for which \( K_c \) is not locally connected. Nevertheless, it is believed that the local connectedness of \( M \) does in fact hold. This would have important consequences one of which is discussed in Special Section 5.

Yet another characterization of \( M \) has recently been given by F.v. Haeseler [Ha]. Using the coordinate change \( z = 1/u \) one first transforms \( p_c \) into the rational mapping \( R_c(u) = u^2/(1 + cu^2) \). The superattractive fixed point for all \( c \) is then \( u = 0 \), and in a neighborhood of \( 0 \) \( R_c \) can be conjugated to \( R_0 \) (only for \( c \in M \), of course, can this conjugation be extended to the entire basin of attraction \( A(0) \)). Let \( \Phi_c(u) = u + a_2(c)u^2 + a_3(c)u^3 + \ldots \) be that local conjugation. Then

\[
M = \{ c \in \mathbb{C} : |a_k(c)| < k, k = 2, 3, \ldots \}.
\]

This bears an intriguing relationship to the Bieberbach conjecture which was recently proved by L. de Branges [Br]. Let

\[
S = \{ f : D \to \mathbb{C} : f(x) = x + a_2x^2 + \ldots, f \text{ analytic and injective} \},
\]

where \( D \) is the open unit disk; the functions in \( S \) are called schlicht functions. The Bieberbach conjecture was:

\[
|f_s| < k, k = 2, 3, \ldots
\]

As a consequence of (4.9) F.v. Haeseler obtained

\[
M \subset \{ c \in \mathbb{C} : |c| < 2 \}.
\]

Since \( c = -2 \) belongs to \( M \), the estimate could not be better.