Chapter 3. The Geodesic Flow

In this chapter we are going to introduce a new aspect of a closed geodesic. Whereas in the previous chapters a closed geodesic was considered as a closed curve distinguished in the space of all closed curves by a certain property (i.e. being a critical value of a functional), we are now going to view a closed geodesic (or rather the tangent vector field along a closed geodesic) as a periodic orbit in the geodesic flow on the tangent bundle of a Riemannian manifold.

The geodesic flow is a special case of a Hamiltonian flow. This observation will put at our disposal the extensive theory of Hamiltonian systems, with particular attention being paid to periodic orbits in such a system.

In section one we give a brief account of the theory of Hamiltonian systems with special emphasis on the geodesic flow. The second section is devoted to the index theorem, whereas in section three we study generic properties of the geodesic flow. A major point here is the Birkhoff-Lewis Fixed Point Theorem. In an appendix we present Moser’s proof of this theorem for the differentiable case.

3.1 Hamiltonian Systems

In this section we recollect the basic theory of Hamiltonian systems and then introduce the geodesic flow as a particular system of this sort.

A symplectic manifold is an even-dimensional manifold $N$ endowed with a closed 2-form $\alpha$ of maximal rank, i.e. $d\alpha = 0$, $\alpha \neq 0$ if dim $N = 2n$.

Darboux’s Lemma implies the existence of a symplectic atlas on a symplectic manifold $(N, \alpha)$. That is to say, in the charts $(\phi, U)$ of such an atlas with $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ as coordinates in $\phi(U)$, $\alpha$ is represented by the canonical symplectic form over $\mathbb{R}^{2n}$:

$$\alpha = \phi^* \left( \sum_{i=1}^n dx^i \wedge dy^i \right).$$

It follows that the coordinate transformations $\phi' \circ \phi^{-1}$ are canonical or symplectic transformations. That is to say, if $\phi' \circ \phi^{-1}$ is represented by

$$x'^i = x'^i(x, y) \quad y'^i = y'^i(x, y)$$
then \( \sum_i dx'^i \wedge dy'^i = \sum_i dx^i \wedge dy^i \), i.e. the functional matrix

\[
d(\phi' \circ \phi^{-1}) = \begin{pmatrix}
\frac{\partial x'^i}{\partial x^j} & \frac{\partial x'^i}{\partial y^j} \\
\frac{\partial y'^i}{\partial x^j} & \frac{\partial y'^i}{\partial y^j}
\end{pmatrix}
\]

is symplectic:

\[
d(\phi' \circ \phi^{-1}) \circ J \circ d(\phi' \circ \phi^{-1}) = J
\]

where

\[
J = \begin{pmatrix}
0_n & E_n \\
-E_n & 0_n
\end{pmatrix}
\]

The most important example of the natural occurrence of a symplectic manifold is the cotangent bundle:

**3.1.1 Proposition.** The cotangent bundle \( T^*M \) of a differentiable (Euclidean) manifold carries a canonical symplectic structure \( \omega \) which is defined as follows.

Consider the diagram of bundle maps

\[
\begin{array}{ccc}
TT^*M & \rightarrow & T^*M \\
\tau_{T^*M} & & \tau_{T^*M} \\
T\tau_{M}^* & \rightarrow & M \\
\tau_M & & \tau_M
\end{array}
\]

Define a 1-form \( \theta \) on \( T^*M \) by \( \theta: T^*xT^*M \rightarrow \mathbb{R}; \xi \mapsto \omega . T\tau_{M}^*\xi \) and define \( \alpha \) by \( -d\theta \).

**Proof.** In a local representation this diagram has the form

\[
(x, y, \xi, \eta) \rightarrow (x, y) \\
\downarrow \\
(x, \xi) \rightarrow (x)
\]

Here we write, as usual, \( x^i \) instead of \( x^i \circ \tau_M^* \) for the coordinate on \( T^*M \). Now \( \theta \) is represented by \( \sum y^i dx^i \). Hence, \( \alpha \) is represented by \( \sum_i dx^i \wedge dy^i \), i.e. a closed 2-form of maximal rank. \( \square \)

Note that the natural atlas on \( T^*M \) is a symplectic (or canonical) atlas for the symplectic form \( \alpha = -d\theta \).