Ordinary Homology Theory

By an ordinary homology theory $k_\ast$ we shall mean one with $k_n(S^0) = 0$ unless $n = 0$. If $k_0(S^0) = G$, then $k_\ast$ will be called an ordinary homology theory with coefficients $G$. Reduced singular homology $\tilde{H}_\ast(-; G)$ is an ordinary homology theory with coefficients $G$ on the category $\mathcal{P}\mathcal{F}'$. We shall show that any two ordinary homology theories with coefficients $G$ satisfying the wedge and WHE axioms are naturally equivalent. We shall also construct the Eilenberg–MacLane spectrum $H(G)$ with

$$\pi_n(H(G)) = \begin{cases} G & n = 0 \\ 0 & n \neq 0. \end{cases}$$

The resulting homology theory $H(G)_\ast(-)$ is then an ordinary homology theory with coefficients $G$, so that on $\mathcal{P}\mathcal{F}'$ we have a natural equivalence $H(G)_\ast(-) \simeq \tilde{H}_\ast(-; G)$. Finally we establish a sufficient condition for the Hurewicz homomorphism $h: \pi_n(Y, y_0) \to H_n(Y; \mathbb{Z})$ (cf. 7.39) to be an isomorphism.

10.1. For each $n \geq 0$ we take the standard $n$-simplex $\Delta_n \subset \mathbb{R}^{n+1}$ to be the simplex with vertices $e_0 = (1,0,\ldots,0), e_1 = (0,1,\ldots,0), \ldots, e_n = (0,\ldots,0,1)$. For each $i, 0 \leq i \leq n$, let $\partial_i: \Delta_{n-1} \to \Delta_n$ be the map defined by

$$\partial_i(e_k) = \begin{cases} e_k & k < i \\ e_{k+1} & k \geq i. \end{cases}$$

on the vertices and extended to be affine linear, i.e.

$$\partial_i(\sum_{j=0}^{n-1} \lambda_j e_j) = \sum_{j=0}^{n-1} \lambda_j \partial_i(e_j) \quad \text{for all } \lambda_0, \ldots, \lambda_{n-1} \text{ with } \lambda_j \geq 0, 0 \leq j \leq n - 1, \sum_{j=0}^{n-1} \lambda_j = 1.$$

A singular $n$-simplex in a topological space $X$ is a map $u: \Delta_n \to X$. The $i$th face $\partial_i u$ of $u$ is the singular $(n - 1)$-simplex $u \circ \partial_i: \Delta_{n-1} \to X$.

Let $S_n(X)$ is the free abelian group generated by the singular $n$-simplices of $X$. $S_n(X)$ is called the group of $n$-chains of $X$. Let $d: S_n(X) \to$
$S_{n-1}(X)$ be the homomorphism defined by $du = \sum_{i=0}^{n} (-1)^i \partial_i u$ on the basis. $d$ is called the differential. One verifies that $d \circ d = 0$, so that \{${S_n(X),d}$\} becomes a chain complex. We define

\[Z_n(X) = \ker \{d: S_n(X) \to S_{n-1}(X)\} \subseteq S_n(X)\]
\[B_n(X) = \text{im} \{d: S_{n+1}(X) \to S_n(X)\} \subseteq S_n(X)\].

$Z_n(X)$ is the group of cycles, $B_n(X)$ the group of boundaries. Since $d \circ d = 0$, $B_n(X) = Z_n(X)$. The singular homology group $H_n(X)$ is defined to be $H_n(X) = Z_n(X)/B_n(X)$, $n \geq 0$, $= 0$ for $n < 0$.

If $f: X \to Y$ is a map, then we can define a homomorphism $f_*: S_n(X) \to S_n(Y)$ for $n \geq 0$ by taking $f_*(u) = f \circ u: A_n \to Y$ on the basis. Then

\[d \circ f_*(u) = \sum_{i=0}^{n} (-1)^i (f \circ u) \circ \partial_i = \sum_{i=0}^{n} (-1)^i f_*(\partial_i u)\]
\[= f_* (\sum_{i=0}^{n} (-1)^i \partial_i u) = f_*(du)\]

for all singular simplices $u$. Hence $d \circ f_* = f_\# \circ d$—i.e. $f_*: \{S_*(X),d\} \to \{S_*(Y),d\}$ is a chain map. Since

\[f_* (Z_n(X)) \subseteq Z_n(Y)\]
\[f_* (B_n(X)) \subseteq B_n(Y)\]

$f_\#$ induces a homomorphism $f_*: H_n(X) \to H_n(Y)$ for all $n \in \mathbb{Z}$.

If $(X,A) \in \mathcal{F}$ we define $S_n(X,A)$ to be the quotient $S_n(X)/S_n(A)$. $d: S_n(X) \to S_{n-1}(X)$ induces a differential $d$ on $S_*(X,A)$, and we define

\[H_n(X,A) = \ker d/im d = \begin{cases} Z_n(X,A)/B_n(X,A) & n \geq 0, \\ 0 & n < 0. \end{cases}\]

A map $f: (X,A) \to (Y,B)$ defines $f_*: (S_n(X),S_n(A)) \to (S_n(Y),S_n(B))$ which in turn induces $f_*: S_n(X,A) \to S_n(Y,B)$. $f_\#$ induces a homomorphism $f_*: H_n(X,A) \to H_n(Y,B)$, $n \in \mathbb{Z}$.

If

\[0 \to A_\# \xrightarrow{f_\#} B_\# \xrightarrow{g_\#} C_\# \to 0\]

is an exact sequence of chain complexes and chain maps then we can construct a homomorphism $\partial_n: H_n(C_\#) \to H_{n-1}(A_\#)$ for every $n$ as follows: given $z \in Z_n(C_\#)$ we can find an $x \in B_n$. With $g_n(x) = z$. Then $g_{n-1}(d_B x) = d_B g_n(x) = d_B z = 0$, so by exactness there is a $y \in A_{n-1}$ with $f_{n-1}(y) = d_B x$. Then $f_{n-2}(d_A y) = d_B f_{n-1}(y) = d_B \circ d_B x = 0$, so since $f_\#$ is a monomorphism $d_A y = 0$—i.e. $y \in Z_{n-1}(A)$. We define $\partial_n(z) = \{y\}$ and check that $\partial_n$ is well defined and a homomorphism. $\partial_n$ is also natural with respect to morphisms of exact sequences: