Chapter XI

Simple algebras over A-fields

§ 1. Ramification. In this Chapter, \( k \) will be an A-field; we use all the notations introduced for such fields in earlier Chapters, such as \( k_A, k_v, r_v, \) etc. We shall be principally concerned with a simple algebra \( A \) over \( k \); as stipulated in Chapter IX, it is always understood that \( A \) is central, i.e. that its center is \( k \), and that it has a finite dimension over \( k \); by corollary 3 of prop. 3, Chap. IX-1, this dimension can then be written as \( n^2 \), where \( n \) is an integer \( \geq 1 \). We use \( A_v \), as explained in Chapters III and IV, for the algebra \( A_v = A \otimes k_v \) over \( k_v \), where, in agreement with Chapter IX, it is understood that the tensor-product is taken over \( k \). By corollary 1 of prop. 3, Chap. IX-1, this is a simple algebra over \( k_v \); therefore, by tho 1 of Chap. IX-1, it is isomorphic to an algebra \( M_{m(v)}(D(v)) \), where \( D(v) \) is a division algebra over \( k_v \); the dimension of \( D(v) \) over \( k_v \) can then be written as \( d(v)^2 \), and we have \( m(v)d(v) = n \); the algebra \( D(v) \) is uniquely determined up to an isomorphism, and \( m(v) \) and \( d(v) \) are uniquely determined. One says that \( A \) is unramified or ramified at \( v \) according as \( A_v \) is trivial over \( k_v \) or not, i.e. according as \( d(v) = 1 \) or \( d(v) > 1 \).

**Theorem 1.** Let \( A \) be a simple algebra over an A-field \( k \); let \( \alpha \) be a finite subset of \( A \), containing a basis of \( A \) over \( k \). For each finite place \( v \) of \( k \), call \( \alpha_v \) the \( r_v \)-module generated by \( \alpha \) in \( A_v \). Then, for almost all \( v \), \( A_v \) is trivial over \( k_v \) and \( \alpha_v \) is a maximal compact subring of \( A_v \).

By corollary 1 of th. 3, Chap. III-1, we may assume that \( \alpha \) is a basis of \( A \) over \( k \), and that \( 1_A \) belongs to it. Call \( \tau \) the reduced trace in \( A \); by prop. 6 of Chap. IX-2, it is not 0, and its \( k_v \)-linear extension to \( A_v \) is the reduced trace in \( A_v \). By lemma 3 of Chap. III-3, we may identify the underlying vector-space of \( A \) over \( k \) with its algebraic dual by putting \( [x,y] = \tau(xy) \). Now, as in th. 3 of Chap. IV-2, take a “basic character” \( \chi \) of \( k_A \). By corollary 1 of that theorem, \( \chi_v \) is of order 0 for almost all \( v \); by corollary 3 of the same theorem, the \( k_v \)-lattice \( \alpha_v \) is its own dual for almost all \( v \), when \( A_v \) is identified with its topological dual by putting \( \langle x,y \rangle = \chi_v(\tau(xy)) \). By corollary 2 of th. 3, Chap. III-1, \( \alpha_v \) is a compact subring of \( A_v \) for almost all \( v \). Therefore, at almost all places \( v \) of \( k \), the assumptions of corollary 2 of prop. 5, Chap. X-2, are valid, the conclusion being as stated in our theorem.
§ 2. The zeta-function of a simple algebra. Let all notations be as in § 1, and let \( \alpha \) be a basis of \( A \) over \( k \). By th. 1 of § 1, \( \alpha_v \) is a maximal compact subring of \( A_v \) for almost all \( v \); therefore we may, for each finite place \( v \) of \( k \), choose a maximal compact subring \( R_v \) of \( A_v \), in such a way that \( R_v = \alpha_v \) for almost all \( v \); that being done, call \( \Phi_v \) the characteristic function of \( R_v \). For each infinite place \( v \) of \( k \), choose an isomorphism of \( A_v \) with \( M_{m(v)}(D(v)) \), where \( D(v) \) is \( R, H \) or \( C \), as the case may be; identifying \( A_v \) with the latter algebra by means of that isomorphism, define \( \Phi_v \) on \( A_v \) by putting, for all \( x \in A_v \), 
\[
\Phi_v(x) = \exp(-\pi \delta \tau(t\tilde{\alpha} \cdot x)),
\]
where notations are the same as in prop. 8 of Chap. X-3. Then \( \Phi = \prod \Phi_v \) is a standard function on \( A^*_A \). Taking now a Haar measure \( \mu \) on \( A^*_A \), we have:

**Proposition 1.** The integral

\[
Z_A(s) = \int_{A^*_A} \Phi(z) |\psi(z)|_A^s d\mu(z)
\]

is absolutely convergent for \( \text{Re}(s) > n \) and is then given by the formula

\[
Z_A(s) = C \prod_{i=0}^{n-1} Z_k(s-i) \prod_v \left( \prod_{0 < h < n \atop h \neq 0(d(v))} \left( 1 - q_v^{h-1} \right) \right) \left( \prod_{0 < h < n \atop h \neq 0(2)} (s-h) \right)^\rho,
\]

where \( Z_k \) is the function defined in theorem 3 of Chap. VII-6, or the zeta-function of \( k \), according as \( k \) is of characteristic 0 or not, where \( \rho \) is the number of real places \( v \) of \( k \) for which \( D(v) = H \), and \( C \) is a constant > 0.

For each \( v \), choose a Haar measure \( \mu_v \) on \( A^*_v \), so that \( \mu_v(R_v) = 1 \) for all finite places \( v \) of \( k \); we may then assume that we have taken \( \mu = \prod \mu_v \), in the same sense as has been explained in Chap. VII-4 for the case \( A = k \). By following step by step the proof of prop. 10, Chap. VII-4, one finds that the integral \( Z_A(s) \) is absolutely convergent, and equal to the infinite product

\[
\prod_v \left( \int_{A_v} \Phi_v(x) |\psi(x)|_v^s d\mu_v(x) \right),
\]

whenever the factors in that product, and the product itself, are absolutely convergent. Those factors have been calculated in propositions 7 and 8 of Chap. X-3; the absolute convergence of \( Z_A(s) \) for \( \text{Re}(s) > n \) is then an immediate consequence of the latter results, combined with prop. 1 of Chap. VII-1. The same results, combined with the definitions in Chap. VII-6, give now the final formula in our proposition for the case \( A = M_n(k) \); combining this with the corollaries of propositions 7 and 8 of Chap. X-3, one obtains at once the general case of the same formula.