Chapter XII

Local classfield theory

§ 1. The formalism of classfield theory. The purpose of classfield theory is to give a description of the abelian extensions of the types of fields studied in this book, viz., local fields and A-fields. Here we assemble part of the formal machinery common to both types.

**Lemma 1.** Let \( G = G_1 \times N \) be a quasicompact group, \( G_1 \) being compact and \( N \) isomorphic to \( \mathbb{R} \) or \( \mathbb{Z} \); let \( H \) be an open subgroup of \( G \). Then, if \( H \) is contained in \( G_1 \) (i.e. if it is compact), \( N \) is isomorphic to \( \mathbb{Z} \), and \( H \) is of finite index in \( G_1 \); otherwise it is of finite index in \( G \).

Put \( H_1 = H \cap G_1 \); as this is open in \( G_1 \), and \( G_1 \) is compact, it is of finite index in \( G_1 \); this proves the first assertion. As \( H \cap N \) is an open subgroup of \( N \), it is \( N \) if \( N \) is isomorphic to \( \mathbb{R} \); therefore \( H = H_1 \times N \) in that case, and \( G/H \) is isomorphic to \( G_1/H_1 \). If \( N \) is isomorphic to \( \mathbb{Z} \), let \( n_1 \) be a generator of \( N \); if \( H \) is not contained in \( G_1 \), it has an element of the form \( g_1 n_1^v \) with \( g_1 \in G_1, \mu \in \mathbb{Z}, \mu \neq 0 \). As \( G_1/H_1 \) is finite, there is \( v \neq 0 \) such that \( g_1^v \in H_1 \). Then \( n_1^v \) is in \( H \), so that \( H \) contains the group \( H' \) generated by \( H_1 \) and \( n_1^v \). As \( H' \) is obviously of finite index in \( G \), this proves the lemma. Theorem 1 of Chap. IV-4 may be regarded as the special case where \( G = k_x^\times/k_x, H \) being the image of \( \Omega(P) \) in \( k_x^\times/k_x^\times \).

**Lemma 2.** Let \( G = G_1 \times N \), \( G' = G'_1 \times N' \) be quasicompact groups, \( G_1 \) and \( G'_1 \) being compact and \( N, N' \) isomorphic to \( \mathbb{R} \) or \( \mathbb{Z} \). Let \( F \) be a morphism of \( G' \) into \( G \) but not into \( G_1 \). Then \( F^{-1}(G_1) = G'_1 \); the kernel of \( F \) is compact; \( F(G') \) is closed in \( G \), and \( G/F(G') \) is compact.

As \( G_1 \) is the maximal compact subgroup of \( G \), \( F(G_1') \) is contained in \( G_1 \). For \( n' \in N' \), call \( f(n') \) the projection of \( F(n') \) onto \( N \) in \( G \); \( f \) is then a non-trivial morphism of \( N' \) into \( N \), hence, obviously, an isomorphism of \( N' \) onto a closed subgroup of \( N \) with compact quotient; our first and second assertions follow from this at once. We also see now that \( F \) induces on \( N' \) an isomorphism of \( N' \) onto \( F(N') \), and that \( F(N') \cap G_1 = \{1\} \); therefore \( F(G') \) is the direct product of \( F(G'_1) \) and \( F(N') \) and is closed. Finally, \( G/G_1, F(N') \) is clearly isomorphic to \( N/f(N') \), hence compact; as the kernel of the obvious morphism of \( G/F(G') \) onto \( G_1 F(N') \) is the image of \( G_1 \) in \( G/F(G') \), hence compact, \( G/F(G') \) must also be compact.
From now on, in this §, we will consider a field \( K \); later on, this will be either a local field or an \( \mathbf{A} \)-field. As in Chapter IX, we write \( \bar{K} \) for an algebraic closure of \( K \), \( K_{\text{sep}} \) for the union of all separable extensions of \( K \) contained in \( \bar{K} \), and \( \mathfrak{G} \) for the Galois group of \( K_{\text{sep}} \) over \( K \), topologized as usual. We will write \( K_{\text{ab}} \) for the maximal abelian extension of \( K \) contained in \( \bar{K} \); this is the same as the union of all abelian extensions of \( K \) of finite degree, contained in \( \bar{K} \), i.e. of all the Galois extensions of \( K \) of finite degree, contained in \( \bar{K} \), whose Galois group is commutative; by definition, this is contained in \( K_{\text{sep}} \). We denote by \( \mathfrak{G}^{(1)} \) the subgroup of \( \mathfrak{G} \) corresponding to \( K_{\text{ab}} \); this is the smallest closed normal subgroup of \( \mathfrak{G} \) such that \( \mathfrak{G}/\mathfrak{G}^{(1)} \) is commutative; it is therefore the same as the "topological commutator-group" of \( \mathfrak{G} \), i.e. the closure of the subgroup of \( \mathfrak{G} \) generated by the commutators of elements of \( \mathfrak{G} \). We write \( \mathfrak{A} \) for the Galois group of \( K_{\text{ab}} \) over \( K \); this may be identified with \( \mathfrak{G}/\mathfrak{G}^{(1)} \); it is a compact commutative group. Let \( \chi \) be any character of \( \mathfrak{G} \); as in Chap. IX-4, call \( \mathfrak{S} \) its kernel and \( L \) the subfield of \( K_{\text{sep}} \) corresponding to \( \mathfrak{S} \), which is the cyclic extension of \( K \) attached to \( \chi \); clearly \( L \subseteq K_{\text{ab}} \) and \( \mathfrak{S} \supseteq \mathfrak{G}^{(1)} \), so that we may identify \( \chi \) with a character of \( \mathfrak{A} \), for which we will also write \( \chi \). Conversely, every character of \( \mathfrak{A} \) determines in an obvious manner a character of \( \mathfrak{G} \), with which we identify it. Thus the group of characters of \( \mathfrak{G} \), for which we will write \( X_{\mathfrak{G}} \), is identified with the group of characters of \( \mathfrak{A} \); the latter is the same as the dual \( \mathfrak{A}^* \) of \( \mathfrak{A} \), except that we will always write the group \( X_{\mathfrak{A}} \) multiplicatively; we put on \( X_{\mathfrak{A}} \) the discrete topology, this being in agreement with the fact that the dual of a compact commutative group is always discrete. By the duality theory, the intersection of the kernels of all the characters of \( \mathfrak{A} \) is the neutral element; this is the same as to say that the intersection of the kernels \( \mathfrak{S} \) of all the characters \( \chi \) of \( \mathfrak{G} \) is \( \mathfrak{G}^{(1)} \), or also that \( K_{\text{ab}} \) is generated by all the cyclic extensions \( L \) of \( K \); this is of course well-known.

Let \( K' \) be any field containing \( K \); as in Chap. IX-3, we take an algebraic closure \( \bar{K}' \) of \( K' \) and assume at the same time that we have taken for \( \bar{K} \) the algebraic closure of \( K \) in \( \bar{K}' \); then, as we have seen there, \( K_{\text{sep}} \) is contained in \( K'_{\text{sep}} \), and, if \( \mathfrak{G}' \) is the Galois group of \( K'_{\text{sep}} \) over \( K' \), the restriction morphism \( \rho \) of \( \mathfrak{G}' \) into \( \mathfrak{G} \) is the one which maps every automorphism of \( K_{\text{sep}} \) over \( K' \) onto its restriction to \( K_{\text{sep}} \). Obviously \( \rho \) maps \( \mathfrak{G}'^{(1)} \) into \( \mathfrak{G}^{(1)} \), so that it determines a morphism of \( \mathfrak{A}' = \mathfrak{G}'/\mathfrak{G}'^{(1)} \) into \( \mathfrak{A} = \mathfrak{G}/\mathfrak{G}^{(1)} \), which we also denote by \( \rho \) and call the restriction morphism of \( \mathfrak{A}' \) into \( \mathfrak{A} \). It amounts to the same to say that \( K_{\text{ab}} \) is contained in \( K'_{\text{ab}} \), and that \( \rho \) maps an element \( \chi' \) of \( \mathfrak{A}' \), i.e. an automorphism of \( K'_{\text{ab}} \) over \( K' \), onto its restriction to \( K_{\text{ab}} \). Correspondingly, \( \chi \mapsto \chi' \circ \rho \) is a morphism of \( X_{\mathfrak{A}} \) into \( X_{\mathfrak{A}'} \).

In classfield theory, one defines a "pairing" of the group \( X_{\mathfrak{A}} \) of the characters of \( \mathfrak{G} \) (or, what amounts to the same, of \( \mathfrak{A} \)) with a locally