Chapter II

Lattices and duality over local fields

§ 1. Norms. In this § and the next one, $K$ will be a $p$-field, commutative or not. We shall mostly discuss only left vector-spaces over $K$; everything will apply in an obvious way to right vector-spaces. Only vector-spaces of finite dimension will occur; it is understood that these are always provided with their “natural topology” according to corollary 1 of th. 3, Chap. I-2. By th. 3 of Chap. I-2, every subspace of such a space $V$ is closed in $V$. Taking coordinates, one sees that all linear mappings of such spaces into one another are continuous; in particular, linear forms are continuous. Similarly, every injective linear mapping of such a space $V$ into another is an isomorphism of $V$ onto its image. As $K$ is not compact, no subspace of $V$ can be compact, except $\{0\}$.

**DEFINITION 1.** Let $V$ be a left vector-space over the $p$-field $K$. By a $K$-norm on $V$, we understand a function $N$ on $V$, with values in $\mathbb{R}^+$, such that:

(i) $N(v)=0$ if and only if $v=0$;
(ii) $N(xv)=\text{mod}_K(x)N(v)$ for all $x \in K$ and all $v \in V$;
(iii) $N$ satisfies the ultrametric inequality

$$N(v+w) \leq \sup(N(v),N(w))$$

for all $v, w$ in $V$.

On $K^n$, one defines a $K$-norm $N_0$ by putting $N_0(x) = \sup_{1 \leq i \leq n} (\text{mod}_K(x_i))$ for all $x = (x_1, \ldots, x_n)$ in $K^n$. As every vector-space of finite dimension over $K$ is isomorphic to a space $K^n$, this shows that there are $K$-norms on all such spaces.

One can obviously use any $K$-norm on $V$ in order to topologize $V$, by taking $N(v-w)$ as distance-function.

**PROPOSITION 1.** Let $V$ be a left vector-space of finite dimension over the $p$-field $K$. Then every $K$-norm on $V$ defines the natural topology on $V$. In particular, every such norm $N$ is continuous, and the subsets $L_r$ of $V$ defined by $N(v) \leq r$ are compact neighborhoods of $0$ for all $r > 0$.

As to the first assertion, in view of corollary 1 of th. 3, Chap. I-2, we need only show that the topology defined by $N$ on $V$ makes $V$ into a topological vector-space over $K$. This follows at once from the inequality

$$N(x'v'-xv) \leq \sup(\text{mod}_K(x')N(v'-v), \text{mod}_K(x'-x)N(v))$$

which is an immediate consequence of def. 1. Therefore $N$ is continuous, and the sets $L_r$ make up a fundamental system of closed neighborhoods.
of 0; in particular, \( L_r \) must be compact for some \( r > 0 \). Now, for any \( s > 0 \), take \( a \in K^\times \) such that \( \text{mod}_K(a) \leq r/s \); then, as one sees at once, \( L_s \) is contained in \( a^{-1} L_r \); therefore it is compact.

**Corollary 1.** There is a compact subset \( A \) of \( V - \{0\} \) which contains some scalar multiple of every \( v \) in \( V - \{0\} \).

Call \( q \) the module of \( K \), and take a \( K \)-norm \( N \) in \( V \). If \( \pi \) is a prime element of \( K \), we have \( \text{mod}_K(\pi) = q^{-1} \), by th. 6 of Chap. I-4, hence \( N(\pi^n v) = q^{-n} N(v) \) for all \( n \in \mathbb{Z} \) and all \( v \in V \). Let \( A \) be the subset of \( V \) defined by \( q^{-1} \leq N(v) \leq 1 \); by proposition 1, it is compact; and, for every \( v \neq 0 \), one can choose \( n \in \mathbb{Z} \) so that \( \pi^n v \in A \).

Corollary 1 implies the fact that the “projective space” attached to \( V \) is compact.

**Corollary 2.** Let \( \varphi \) be any continuous function on \( V - \{0\} \), with values in \( \mathbb{R} \), such that \( \varphi(av) = \varphi(v) \) for all \( a \in K^\times \) and all \( v \in V - \{0\} \). Then \( \varphi \) reaches its maximum at some point \( v_1 \) of \( V - \{0\} \).

In fact, this will be so if we take \( A \) as in corollary 1 and take for \( v_1 \) the point of \( A \) where \( \varphi \) reaches its maximum on \( A \).

**Corollary 3.** Let \( f \) be any linear form on \( V \), and \( N \) a \( K \)-norm on \( V \). Then there is \( v_1 \neq 0 \) in \( V \), such that

\[
N(v)^{-1} \text{mod}_K(f(v)) \leq N(v_1)^{-1} \text{mod}_K(f(v_1))
\]

for all \( v \neq 0 \) in \( V \).

This is a special case of corollary 2, that corollary being applied to the left-hand side of (2). If one denotes by \( N^*(f) \) the right-hand side of (2), then \( N^*(f) \) is the smallest positive number such that

\[
\text{mod}_K(f(v)) \leq N^*(f) \cdot N(v)
\]

for all \( v \in V \), and \( f \to N^*(f) \) is a \( K \)-norm on the dual space of \( V \), i.e. on the right vector-space made up of the linear forms on \( V \) (where the addition is the obvious one, and the scalar multiplication is defined by putting \( (fa)(v) = f(v)a \) when \( f \) is such a form, and \( a \in K \)).

By a hyperplane in \( V \), one understands a subspace of \( V \) of codimension 1, i.e. any subset \( H \) of \( V \) defined by an equation \( f(v) = 0 \), where \( f \) is a linear form other than 0; when \( H \) is given, \( f \) is uniquely determined up to a scalar factor other than 0. Now, if (2) is valid for all \( v \neq 0 \), and for a given norm \( N \), a given linear form \( f \neq 0 \) and a given \( v_1 \neq 0 \), it remains so if one replaces \( f \) by \( fa \), with \( a \in K^\times \), and \( v_1 \) by \( bv_1 \) with \( b \in K^\times \); in other words, its validity for all \( v \neq 0 \) depends only upon the hyperplane \( H \) defined by \( f = 0 \) and the subspace \( W \) of \( V \) generated by \( v_1 \); when it holds for all \( v \neq 0 \), we shall say that \( H \) and \( W \) are \( N \)-orthogonal to each other.