Chapter VIII

Traces and norms

§ 1. Traces and norms in local fields. In §§ 1–3, we will consider exclusively local fields (assumed to be commutative). We denote by $K$ a local field and by $K'$ an algebraic extension of $K$ of finite degree $n$ over $K$. If $K$ is an $R$-field and $K' \neq K$, we must have $K = R$, $K' = C$, $n = 2$; then, by corollary 3 of prop. 4, Chap. III-3, $Tr_{C/R}(x) = x + \bar{x}$ and $N_{C/R}(x) = x\bar{x}$; $Tr_{C/R}$ maps $C$ onto $R$, and $N_{C/R}$ maps $C^*$ onto $R_\times^*$, which is a subgroup of $R^\times$ of index 2.

From now on, until the end of § 3, we assume $K$ to be a $p$-field and adopt our usual notations for such fields, denoting by $q$ the module of $K$, by $R$ its maximal compact subring, by $P$ the maximal ideal in $R$, and by $\pi$ a prime element of $K$. The field $K'$ being as stated above, we adopt similar notations, viz., $q'$, $R'$, $P'$, $\pi'$, for $K'$. We write $f$ for the modular degree of $K'$ over $K$ and $e$ for the order of ramification of $K'$ over $K$, as defined in def. 4 of Chap. I-4; then $q' = q^f$ and $n = ef$, by corollary 6 of th. 6, Chap. I-4. As $e = \text{ord}_K(\pi)$, the $R'$-module generated in $K'$ by $P' = \pi' R$, for any $v \in \mathbb{Z}$, is $P'^v$; for this, we will write $i(P^v)$.

By corollary 1 of prop. 4, Chap. III-3, and the remarks following that proposition, $Tr_{K'/K}$ is $\pm 0$ if and only if $K'$ is separable over $K$; then, being $K$-linear, it maps $K'$ onto $K$. By the definition of the norm, and by corollary 3 of th. 3, Chap. I-2, we have, for all $x' \in K'$:

\[ \text{mod}_K(x') = \text{mod}_K(N_{K'/K}(x')). \]

In view of th. 6 of Chap. I-4, this implies that $x' \in R'$ if and only if $N_{K'/K}(x') \in R$, and $x' \in R^\times$ if and only if $N_{K'/K}(x') \in R^\times$. As $\text{mod}_K(\pi) = q^{-1}$ and $\text{mod}_K(\pi') = q^{-f}$, (1) may also be written as follows, for $x' \neq 0$:

\[ \text{ord}_K(N_{K'/K}(x')) = f \cdot \text{ord}_K(x'). \]

From now on, we will write $Tr$, $N$ instead of $Tr_{K'/K}$, $N_{K'/K}$, except when there are more fields to be considered than $K$ and $K'$. For every $v \in \mathbb{Z}$, we will write $\mathfrak{N}(P'^v) = P'^v$; by (2), this is the $R$-module generated in $K$ by the image of $P'^v$ under $N$.

**Proposition 1.** Let $K'$ be separable over $K$. Then, if $x' \in R'$, $Tr(x') \in R$; if $x' \in P'$, $Tr(x') \in P$ and $N(1 + x') = 1 + Tr(x') + y$ with $y \in R \cap x'^2 R'$.
Let $\bar{K}$ be an algebraic closure of $K'$; call $\lambda_1, \ldots, \lambda_n$ the distinct $K$-linear isomorphisms of $K'$ into $\bar{K}$; then, by corollary 3 of prop. 4, Chap. III-3, we have

$$\text{Tr}(x') = \sum_i \lambda_i(x'), \quad N(1 + x') = \prod_i (1 + \lambda_i(x')).$$

Call $K''$ the compositum of the fields $\lambda_i(K')$, which is the smallest Galois extension of $K$ in $\bar{K}$, containing $K'$; define $R''$, $P''$ for $K''$ as $R$, $P$ are defined for $K$. By corollary 5 of th. 6, Chap. I-4, we have $\lambda_i(R') \subset R''$ and $\lambda_i(P') \subset P''$ for all $i$, so that $\text{Tr}(x')$ is in $R''$ if $x' \in R'$, and in $P''$ if $x' \in P'$; as the same corollary shows that $R = K \cap R''$ and $P = K \cap P''$, this proves our assertions concerning $\text{Tr}$. Now assume $x' \in R'$, $x' \neq 0$, and put

$$y = N(1 + x') - 1 - \text{Tr}(x');$$

by (3), this is a sum of monomials of degree $\geq 2$ in the $\lambda_i(x')$. As one of the $\lambda_i$ is the identity, and as the $\lambda_i$, by corollary 2 of prop. 3, Chap. III-2, differ from one another only by automorphisms of $K''$ over $K$, all the $\lambda_i(x')$ have the same order as $x'$ in $K''$, so that $yx'^{-2}$ is in $R''$ if $x'$ is in $R'$. As $R' = K' \cap R''$, this proves our last assertion. In view of the fact that $\text{Tr} = 0$ if $K'$ is inseparable over $K$, and of the remarks about that case in Chap. III-3, our proposition is still valid (but uninteresting) in the inseparable case.

**COROLLARY.** If $x' \in P'^{-e+1}$, $\text{Tr}(x') \in R$.

By definition, $e = \text{ord}_K(\pi)$; therefore our assumption amounts to $\pi x' \in P'$, which implies $\text{Tr}(\pi x') \in P$ by prop. 1, hence $\text{Tr}(x') \in R$ since $\text{Tr}$ is $K$-linear.

**DEFINITION 1.** Let $K'$ be separable over $K$; let $d$ be the largest integer such that $\text{Tr}(x') \in R$ for all $x' \in P'^{-d}$. Then $P^d$ is called the different of $K'$ over $K$, and $d$ its differential exponent.

For the different, we will write $D(K'/K)$, or simply $D$. If $K'$ is inseparable over $K$, $\text{Tr}$ is 0, so that it maps $P'^{-\infty}$ into $R$ for all $v$; in that case we put $d = + \infty$, $D(K'/K) = 0$.

By the corollary of prop. 1, we have $d \geq e-1$. In particular, if $d = 0$, $e = 1$, so that $K'$ is unramified over $K$. The converse is also true; this will be a consequence of the following results:

**PROPOSITION 2.** Let $K'$ be unramified over $K$; call $\rho$, $\rho'$ the canonical homomorphisms of $R$ onto $k = R/P$, and of $R'$ onto $k' = R'/P'$, respectively. Then, for $x' \in R'$, we have

$$\rho(\text{Tr}(x')) = \text{Tr}_{k'/k}(\rho'(x')), \quad \rho(N(x')) = N_{k'/k}(\rho'(x')).$$