II. Isometries of Riemannian Manifolds

1. The Group of Isometries of a Riemannian Manifold

The earliest and very general result on the group of isometries is perhaps the following theorem of van Danzig and van der Waerden [1] (see also Kobayashi-Nomizu [1, vol. 1; pp. 46–50] for a proof).

Theorem 1.1. Let $M$ be a connected, locally compact metric space and $\mathfrak{I}(M)$ the group of isometries of $M$. For each point $x$ of $M$, let $\mathfrak{I}_x(M)$ denote the isotropy subgroup of $\mathfrak{I}(M)$ at $x$. Then $\mathfrak{I}(M)$ is locally compact with respect to the compact-open topology and $\mathfrak{I}_x(M)$ is compact for every $x$. If $M$ is compact, then $\mathfrak{I}(M)$ is compact.

Eleven years later, in 1939, the following result was published by Myers and Steenrod [1].

Theorem 1.2. The group $\mathfrak{I}(M)$ of isometries of a Riemannian manifold $M$ is a Lie transformation group with respect to the compact-open topology. For each $x \in M$, the isotropy subgroup $\mathfrak{I}_x(M)$ is compact. If $M$ is compact, $\mathfrak{I}(M)$ is also compact.

Before we begin the proof, we should perhaps point out that, a priori, there are two definitions of isometry for a Riemannian manifold. A diffeomorphism $f$ of $M$ onto itself is called an isometry if it preserves the metric tensor. We can also call any one-to-one mapping of $M$ onto itself which preserves the distance function defined by the Riemannian metric an isometry of $M$. According to Myers and Steenrod, these two definitions are equivalent (see Kobayashi-Nomizu [1, vol. 1; p. 169] for a proof). In this book, we adopt the first definition.

Let $n = \dim M$. In the original proof of Myers and Steenrod, they took $n+1$ points $x_0, x_1, \ldots, x_n$ which are independent in a certain sense and proved that the mapping $f \in \mathfrak{I}(M) \rightarrow (f(x_0), f(x_1), \ldots, f(x_n)) \in M^{n+1}$ is one-to-one and has a closed submanifold of $M^{n+1}$ as its image. The have proved that the differentiable structure on $\mathfrak{I}(M)$ introduced by the injection $\mathfrak{I}(M) \subset M^{n+1}$ makes $\mathfrak{I}(M)$ into a Lie transformation group. Theorem 1.2 may be also derived immediately from Theorem 3.3.

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(Bochner-Montgomery) of Chapter I and from Theorem 1.1 (van Danzig-van der Waerden). But we prefer to derive it from Theorem 3.2 of Chapter I as follows.

**Proof of Theorem 1.2.** Let $L(M)$ be the bundle of linear frames over $M$; it is a principal bundle with group $GL(n; \mathbb{R})$, $n = \dim M$.

**Lemma 1.** Let $\theta = (\theta^1, \ldots, \theta^n)$ be the canonical form on $L(M)$. For every transformation $f$ of $M$, the induced automorphism $\tilde{f}$ of $L(M)$ leaves $\theta$ invariant. Conversely, every fibre-preserving transformation of $L(M)$ leaving $\theta$ invariant is induced by a transformation of $M$.

**Proof of Lemma 1.** Let $u \in L(M)$ and $X^* \in T_u(L(M))$. We set $X = \pi(X^*) \in T_x(M)$, where $\pi : L(M) \to M$ is the projection and $x = \pi(u)$. Then

$$\theta(X^*) = u^{-1}(X) \quad \text{and} \quad \theta(\tilde{f} X^*) = \tilde{f}(u)^{-1}(f X),$$

where the frames $u$ and $\tilde{f}(u)$ are considered as linear mappings of $\mathbb{R}^n$ onto $T_x(M)$ and $T_{f(x)}(M)$, respectively. It follows from the definition of $\tilde{f}$ that the following diagram is commutative:

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R^n
  u  \downarrow \quad \downarrow f(u)
T_x(M) \quad \rightarrow \quad T_{f(x)}(M).
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Hence, $u^{-1}(X) = \tilde{f}(u)^{-1}(f X)$, thus proving that $\theta$ is invariant by $\tilde{f}$.

Conversely, if $F$ is a fibre-preserving transformation of $L(M)$ leaving $\theta$ invariant, let $f$ be the transformation of the base $M$ induced by $F$. If we set $J = \tilde{f}^{-1} \circ F$, then $J$ is also a fibre-preserving transformation of $L(M)$ leaving $\theta$ invariant and induces the identity transformation on the base $M$. Hence,

$$u^{-1}(X) = \theta(X^*) = \theta(JX^*) = J(u)^{-1}(X) \quad \text{for} \quad X^* \in T_u(L(M)).$$

This implies $J(u) = u$, that is, $\tilde{f}(u) = F(u)$.

Let $\omega = (\omega_i^j)_{i,j = 1, \ldots, n}$ be the connection form for an affine connection of $M$. Then a transformation $f$ of $M$ is an affine transformation if $\tilde{f}$ preserves $\omega$. From Lemma 1, we obtain

**Lemma 2.** Let $\theta$ and $\omega$ be the canonical form and a connection form on $L(M)$ respectively. If $f$ is an affine transformation of $M$, then $\tilde{f}$ preserves both $\theta$ and $\omega$. Conversely, every fibre-preserving transformation of $L(M)$ leaving both $\theta$ and $\omega$ invariant is induced by an affine transformation of $M$.

Lemma 2 implies that the group $\mathfrak{A}(M)$ of affine transformations of $M$ is isomorphic to the group of bundle automorphisms of $L(M)$ leaving both $\theta$ and $\omega$ invariant. On the other hand, the $n + n^2$ 1-forms $\theta = (\theta^i)$ and $\omega = (\omega^i_j)$ define an absolute parallelism, i.e., a $\{1\}$-structure, on