Chapter V. Function Theory for the Subgroups of Finite Index in the Modular Group

In this chapter we introduce functions and forms of integral dimension for subgroups \( \Gamma_1 \) of finite index in \( \Gamma \), and as in chapter II, at first without regard to their existence, we investigate their basic properties. Then again, independent of questions of existence, we focus on modular forms of dimension \(-2\) and their connection with certain integrals. The existence of such functions and forms will be derived from the Riemann-Roch Theorem of the theory of algebraic functions. In order to understand this theorem we compile certain facts. In conclusion we apply the Riemann-Roch Theorem to the calculation of the \( \mathbb{C} \)-dimension of the vector space of entire modular forms of fixed dimension.

§ 1. Functions for Subgroups

The subgroups \( \Gamma_1 \) and \( \Gamma_1 \cup (-1) \Gamma_1 \) support the same functions. We therefore assume in this section that \( -1 \in \Gamma_1 \). We denote the finite index of \( \Gamma_1 \) in \( \Gamma \) by \( \mu \).

1. Definition. A function \( f \) on \( \mathcal{H}^* \) is called a modular function for \( \Gamma_1 \) if the following hold:
   a) \( f \) is meromorphic on \( \mathcal{H} \);
   b) for all \( S \in \Gamma_1 \) and \( \tau \in \mathcal{H}^* \), \( f(S(\tau)) = f(\tau) \);
   c) at a rational cusp \( \frac{d}{c} \), \( (c, d) = 1 \), \( f \) has an expansion of the form

   \[
f(\tau) = \sum_{v \geq v_0} c_v e^{2\pi i \kappa_1 A(\tau)v}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,
   \]

   where \( \kappa_1 \) is a natural number. This expansion is valid for \( \tau \in \mathcal{H} \) with imaginary part of \( A(\tau) \) sufficiently large.

   \( f \left( \frac{-d}{c} \right) \) is defined in the usual way by (1).
The cusp \( \frac{d}{c} \) determines the matrix \( A \) only up to a factor of \( \pm U^n \) with arbitrary \( n \in \mathbb{Z} \) and thus determines \( e^{\frac{2\pi i}{\kappa_1} A(\tau)} \) to within a \( \kappa_1^{-th} \) root of unity. One can choose \( \kappa_1 = \kappa \) the fan width of the fundamental region for \( \Gamma_1 \) at \( -\frac{d}{c} \). This follows from the expansion (1), since

\[ A^{-1} U^k A \in \Gamma_1 \]

and by b) the left side in (1) is invariant under the transformation

\[ \tau \mapsto A^{-1} U^k A(\tau) \quad \text{or} \quad A(\tau) \mapsto U^k A(\tau). \]

2. First properties. We enumerate some properties of modular functions for \( \Gamma_1 \) which follow directly from Definition 1:

1. The modular functions for \( \Gamma_1 \) form a field.
2. If \( f \) is a modular function for \( \Gamma_1 \) and if \( \Gamma_2 \) is a subgroup of finite index in \( \Gamma_1 \), then \( f \) is a modular function for \( \Gamma_2 \).
3. If \( f \) is a modular function for \( \Gamma_1 \) and is invariant under the transformations of the group extension \( \Gamma_2 \subset \Gamma \), then \( f \) is a modular function for \( \Gamma_2 \).
4. If \( f \) is a modular function for \( \Gamma_1 \) and \( S \in \Gamma \), then

\[ f|S: \mathcal{H}^* \rightarrow \hat{\mathbb{C}}, \quad \tau \mapsto f(S(\tau)), \]

is a modular function for \( S^{-1} \Gamma_1 S \).

Concerning 4. we remark that the first two conditions are obviously fulfilled. The validity of c) follows from the existence of an expansion

\[ f(\tau) = \sum_{v \geq v_0} c_v e^{\frac{2\pi i}{\kappa_1} A S^{-1}(\tau)^v} \]

at the point \( S A^{-1}(\infty) \) when \( \tau \) is replaced by \( S(\tau) \).

\[ f|S \text{ coincides formally even for real } S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc > 0, \text{ with } f|_k S \text{ for } k = 0 \text{ as defined in II, §4,1 when } \mathcal{H}^* \text{ replaces } \mathcal{H}. \]

We prove

Theorem 1. Each modular function \( f \) for \( \Gamma_1 \) satisfies an algebraic equation of degree \( \mu \),

\[ G(f) = \sum_{v=0}^{\mu} R_v(J) f^v = 0, \tag{2} \]

where the \( R_v(J) \) are rational functions of \( J \) over \( \mathbb{C} \).