Chapter III. Generalized Theta Functions

§ 7. Theta Factors of Automorphy and Generalized Theta Functions

If $M$ is a marked compact Riemann surface of genus $g > 0$ and $\xi$ is a factor of automorphy with characteristic class $c(\xi) = r$ then by the Riemann-Roch theorem $\gamma(\xi) = \gamma(\kappa \xi^{-1}) + r + 1 - g$, where $\kappa$ is the canonical factor of automorphy; and if $r \geq 2g - 1$ then $c(\kappa \xi^{-1}) = 2g - 2 - r < 0$ so that $\gamma(\kappa \xi^{-1}) = 0$ and $\gamma(\xi) = r + 1 - g$. Thus all factors of automorphy with characteristic class $r$ admit equally many complex analytic relatively automorphic functions whenever $r \geq 2g - 1$. Now the set of factors of automorphy with characteristic class $r$ can be parametrized by the complex manifold $\mathbb{C}^g$, by associating to any point $t \in \mathbb{C}^g$ the factor of automorphy $\rho_t \xi^r$ where $\rho: \mathbb{C}^g \to \text{Hom}(\Gamma, \mathbb{C}^*)$ is the canonical homomorphism considered previously and $\xi$ is a factor of automorphy representing the point bundle associated to the divisor $1 \cdot p_0$ for the base point $p_0$ of the marked surface $M$; and it can be asked whether the set of all complex analytic relatively automorphic functions for these factors of automorphy can be parametrized accordingly by the complex manifold $\mathbb{C}^g \times \mathbb{C}^{r+1-g}$ if $r \geq 2g - 1$. The answer is provided by the following result.

**Theorem 7.** Let $M$ be a marked compact Riemann surface of genus $g > 0$ with universal covering space $\tilde{M}$, and $\xi$ be a factor of automorphy representing the point bundle associated to the divisor $1 \cdot p_0$ where $p_0$ is the base point of $M$. Then if $r \geq 2g - 1$ there are $r + 1 - g$ complex analytic functions $f_i$ on $\mathbb{C}^g \times \tilde{M}$ such that

\[
(1) \quad f_i(t, Tz) = \rho_i(T) \xi(T, z)f_i(t, z) \quad \text{whenever} \quad T \in \Gamma
\]

and such that for each fixed $t \in \mathbb{C}^g$ these are linearly independent functions on $\tilde{M}$ hence form a basis for the space of complex analytic relatively automorphic functions for the factor of automorphy $\rho_i \xi^r$.

**Proof.** The first step in the proof is the demonstration of the local form of the theorem, the assertion that for any point $t_0 \in \mathbb{C}^g$ there exist an open neighborhood $U$ of $t_0$ in $\mathbb{C}^g$ and complex analytic functions $f_i(t, z)$ in $U \times \tilde{M}$ such that for each fixed point $t \in U$ these functions form a basis for the space of complex analytic relatively automorphic functions for the factor of automorphy $\rho_i \xi^r$. For this purpose it is convenient to use local coordinates in $U$ provided by the Jacobi mapping. Choose fixed base points $a_i \in \tilde{M}$ and open sets $\tilde{U}_i \subseteq \tilde{M}$ such that

\[
\tilde{f}_g: \tilde{U}_1 \times \cdots \times \tilde{U}_g \to U
\]
is a complex analytic homeomorphism, where as earlier \( \tilde{\phi}_g(z_1, \ldots, z_g) = \tilde{\phi}(z_1) + \cdots + \tilde{\phi}(z_g) - \tilde{\phi}(a_1) - \cdots - \tilde{\phi}(a_g) \) for any points \( z_i \in \tilde{U}_i \) and \( \tilde{\phi}: \tilde{M} \to \mathbb{C}^g \) is the Jacobi mapping; the sets \( U_i \) can be taken to be disjoint. In these terms the desired local assertion is that there are \( r + 1 - g \) complex analytic functions \( f_i(z_1, \ldots, z_g, z) \) in \( \tilde{U}_1 \times \cdots \times \tilde{U}_g \times \tilde{M} \) such that

\[
f_i(z_1, \ldots, z_g, Tz) = \rho_{\tilde{\phi}_g(z_1, \ldots, z_g)}(T)^{-1} \xi(T, z) f_i(z_1, \ldots, z_g, z) \quad \text{whenever} \quad T \in \Gamma,
\]

and that for each fixed \( (z_1, \ldots, z_g) \) these functions are linearly independent.

For a useful auxiliary construction let \( \zeta_a \) be a factor of automorphy of characteristic class \( g \) representing the line bundle associated to the divisor 1·\( a_1 + \cdots + 1·a_g \); thus \( \zeta_a \) admits a complex analytic relatively automorphic function \( h \) for which \( \mathfrak{d}(h) = 1·a_1 + \cdots + 1·a_g \). Consider then the factor of automorphy \( \zeta_a \zeta_a^r \) of characteristic class \( r + g \geq 3g - 1 \), and note that as a consequence of the Riemann-Roch theorem \( \gamma(\zeta_a \zeta_a^r) = r + 1 \). There are therefore \( r + 1 \) linearly independent complex analytic functions \( h_1, \ldots, h_r \) on \( \tilde{M} \) such that

\[
h_j(Tz) = \zeta_a(T, z) \zeta(T, z)^r h_j(z) \quad \text{whenever} \quad T \in \Gamma.
\]

It further follows from the Riemann-Roch theorem that for any fixed points \( z_i \in \tilde{U}_i \) the \((r + 1) \times g\) matrix \( (h_j(z_i)) \) must be of rank \( g \). Indeed if that matrix had rank \( \rho < g \) there would exist \( r + 1 - \rho > r + 1 - g \) linearly independent vectors \( c^k \in \mathbb{C}^{r+1}, 1 \leq k \leq r + 1 - \rho \), such that \( \sum_{j=1}^{r+1} c_j^k h_j(z_i) = 0 \); hence there would exist \( r + 1 - \rho > r + 1 - g \) linearly independent complex analytic relatively automorphic functions \( \sum_{j=1}^{r+1} c_j^k h_j \) for the factor of automorphy \( \zeta_a \zeta_a^r \), all of which vanish at the points \( z_1, \ldots, z_g \). The points \( z_i \) represent distinct points on \( M \) since the neighborhoods \( U_i \) are disjoint, so it follows readily that there are at least \( r + 1 - \rho > r + 1 - g \) linearly independent complex analytic relatively automorphic functions for the factor of automorphy \( \zeta_a \zeta_a^r \zeta_z^{-1} \) where \( z \) is a factor of automorphy of characteristic class \( g \) representing the line bundle associated to the divisor \( 1·z_1 + \cdots + 1·z_g \); but from the Riemann-Roch theorem \( \gamma(\zeta_a \zeta_a^r \zeta_z^{-1}) = r + 1 - g \), which is a contradiction. The matrix \( (h_j(z_i)) \) is thus of rank \( g \) for any points \( z_i \in \tilde{U}_i \); hence after shrinking the neighborhoods further if necessary this matrix can be taken to consist of the last \( g \) columns of a matrix \( C(z_1, \ldots, z_g)^{-1} \) where \( C: \tilde{U}_1 \times \cdots \times \tilde{U}_g \to GL(r + 1, \mathbb{C}) \) is a complex analytic mapping.

Thus there are complex analytic functions \( c_{ij}(z_1, \ldots, z_g) \) in \( \tilde{U}_1 \times \cdots \times \tilde{U}_g \) such that \( \sum_{j=1}^{r+1} c_{ij}(z_1, \ldots, z_g) h_j(z_i) = 0 \) for \( 1 \leq i \leq r + 1 - g, 1 \leq k \leq g \), and all points \( z_i \in \tilde{U}_i \). The functions

\[
g_j(z_1, \ldots, z_g, z) = \sum_{i=1}^{r+1} c_{ij}(z_1, \ldots, z_g) h_j(z) \quad \text{for} \quad 1 \leq i \leq r + 1 - g
\]

are therefore complex analytic functions on \( \tilde{U}_1 \times \cdots \times \tilde{U}_g \times \tilde{M} \), and for each fixed \( (z_1, \ldots, z_g) \in \tilde{U}_1 \times \cdots \times \tilde{U}_g \) are \( r + 1 - g \) linearly independent relatively automorphic functions for the factor of automorphy \( \zeta_a \zeta_a^r \) and vanish at the points \( \Gamma z_i \).

Returning then to the proof of the local form of the theorem, choose a point \( z_0 \in \tilde{M} \) which is not contained in any of the sets \( \Gamma \tilde{U}_i \) or \( \Gamma a_i \) and consider the