3. The Mechanics of Deformation

3.1 Finite Strain in Rheological Bodies

3.1.1 The Physics of Deformation

The basic problem in the study of the Earth’s crust is to understand its deformations. Unfortunately, the general physics of deformations is not as well understood as one might desire.

Deformations can occur in two fundamentally different ways: viz. continuously or discontinuously. Under continuous deformation we understand a condition where neighboring points in a material always remain neighboring points, whereas in discontinuous displacements this is not the case.

Several branches of the theory of continuous deformation have been very intensively developed. It is well known that the theory of elasticity has been carried to a high degree of refinement; the same is true for the hydrodynamics of viscous fluids. Unfortunately, the materials of which the Earth’s crust is composed are very unlikely to fit either of these theories. Of those branches of the continuous displacement theory that are more or less well developed, the theory of plasticity has the most bearing upon the displacements observed in the Earth’s crust. However, the theory of plasticity has been developed for the description of the behavior of a metal during cold working, and it cannot be expected, therefore, that its application to the Earth’s crust will lead to entirely satisfactory results. The material in the Earth’s interior shows a very complicated behavior, possibly of the types discussed in the various theories of “rheology”. However, these theories all appear to be more or less heuristic and therefore incomplete.

Very important is the discussion of discontinuous displacements. There are many instances where ruptures, fissures, fractures and such like occur in the Earth’s crust. Although humans have been breaking things since the inception of civilization, it is an unfortunate fact that the whole subject of fracture is only very incompletely understood. There are quite a number of rule-of-thumb criteria of fracture or better: of when a structure is supposed to be safe so as not to fracture, — but the basic problem of describing the progress of a fracture surface in a given body under given external stresses has not yet been solved.

We shall, in the following sections, consider the various aspects of the theory of deformations one by one.
3.1.2 The Structure of a Finite Strain Theory

In order to obtain a description of the dynamics of continuous media, there are various steps that must be observed. In the first place, one must decide upon a description of the deformation. Once this has been achieved, one must express various physical laws: the condition of continuity, the law of motion, and boundary conditions. We shall discuss these steps one by one.

**a) Measure of Displacement.** Let us assume that a certain volume \( W \) (which may be infinite) of space is filled with matter. The volume \( W \), and the way in which it is filled, changes with time \( t \).

The points of space may be specified by giving three Cartesian coordinates \( x_i \) such that the line element is defined as follows:

\[
ds^2 = dx_i dx_i .
\]

In this formula, the summation convention has been used, which stipulates that one has to sum over all indices that occur twice.

The above scheme characterizes the geometrical space ("coordinate space") occupied or potentially occupied by the continuous medium. The next task is to characterize the medium. It is well known that this can be done by introducing three parameters \( \xi_a \). The whole medium is characterized if the parameters run through all points of a volume \( \Psi \) in "parameter space", i.e., the space of the \( \xi \)'s.

We shall assume that the space \( \Psi \) of the parameters is endowed with a Cartesian metric, so that a line element \( d\sigma \) can be defined as follows:

\[
d\sigma^2 = d\xi_a d\xi_a .
\]

It is always possible to make such a parameter transformation that the parameters are equal to the coordinates of all the particles at a particular time, say \( t_0 \). It is often convenient to do this.

With the above characterization of a continuous medium and the geometrical space occupied by it, one can proceed to describe motions. The complete motion is obviously given if the geometrical coordinates of each particle of the medium are known for all times:

\[
x_i = x_i(\xi_a, t) .
\]

Thus, the specification of three functions of the three parameters plus time determines the motion. This type of description is often called the *material* form of the description of motion.

One also could have provided a description of the motion in another way, viz. by solving Eq. (3.1.2-3) for the \( \xi \)'s:

---