# Submodular functions and convexity

**L. Lovász**
Eötvös Loránd University, Department of Analysis I, Múzeum krt. 6–8, H-1088 Budapest, Hungary

## 0. Introduction

In "continuous" optimization convex functions play a central role. Besides elementary tools like differentiation, various methods for finding the minimum of a convex function constitute the main body of nonlinear optimization. But even linear programming may be viewed as the optimization of very special (linear) objective functions over very special convex domains (polyhedra). There are several reasons for this popularity of convex functions:

– Convex functions occur in many mathematical models in economy, engineering, and other sciencies. Convexity is a very natural property of various functions and domains occuring in such models; quite often the only non-trivial property which can be stated in general.

– Convexity is preserved under many natural operations and transformations, and thereby the effective range of results can be extended, elegant proof techniques can be developed as well as unforeseen applications of certain results can be given.

– Convex functions and domains exhibit sufficient structure so that a mathematically beautiful and practically useful theory can be developed.

– There are theoretically and practically (reasonably) efficient methods to find the minimum of a convex function.

It is less apperant, but we claim and hope to prove to a certain extent, that a similar role is played in discrete optimization by *submodular set-functions*. These functions do not enjoy the nice geometric image of convex functions, and accordingly their significance has been discovered only gradually. The first class of submodular functions which was studied thoroughly was the class of matroid rank functions. Generalizing certain basic properties of matroid polyhedra, Edmonds (1970) began the systematical study of submodularity. Let us remark, however, that approaching from quite a different angle, Choquet (1955) also introduced these set-functions. He proved that the newtonian "capacity" of a subset of $\mathbb{R}^3$ defines a submodular set-function. Quite a few proof techniques in graph theory, but also in probability, geometry and lattice theory have made implicit use of submodularity of certain set-functions, thus forecasting the importance of this property.

In this paper we survey some of the most important aspects of submodularity. In particular, we shall see that submodularity shares the above-listed four

A. Bachem et al. (eds.), *Mathematical Programming The State of the Art*

important properties of convexity. But besides this formal analogy, we shall develop a more fundamental connection between these two concepts. This connection will enable us to apply some basic facts concerning convex functions to obtain similarly basic results for submodular functions. In particular, a polynomial-time algorithm to find the minimum of a submodular set-function and a "sandwich theorem" for submodular functions can be obtained this way. (A refined integral version of this sandwich theorem, due to A. Frank (1982), needs a direct proof; but this powerful result is also motivated clearly by the convex-submodular analogy.)

Somewhat surprisingly, submodular functions are in some respects also similar to concave functions. This suggests that the maximization problem for submodular functions may also lead to interesting results. We shall see that this problem is substantially more difficult than the minimization problem, and in general no solution exists; but solutions for special cases, as well as reasonable heuristics, can be obtained.

Submodularity gives rise to polyhedra with very nice properties. Following Edmonds (1970), we treat various linear programming problems associated with submodular functions. For a single submodular function, these linear programming problems can be solved by appropriate versions of the *greedy algorithm*. For two submodular functions similar polyhedral considerations lead to very deep min-max results which can be examplified by the Matroid Intersection Theorem.

Several recent combinatorial studies involving submodularity fit into the following pattern. Take a classical graph-theoretical result (e.g. the Marriage Theorem, the Max-flow-min-cut Theorem etc.), and replace certain linear functions occuring in the problem (either in the objective function or in the constraints) by submodular functions. Often the generalizations of the original theorems obtained this way remain valid; sometimes even the proofs carry over. What is important here to realize is that these generalizations are by no means l'art pour l'art. In fact, the range of applicability of certain methods can be extended enormously by this trick. Choosing the submodular functions from among the many submodular functions arising from graphs and other combinatorial structures, various and often surprising results can be obtained this way. As an example, one can obtain, as a "submodular" generalization of the Marriage Theorem, a version of the famous Matroid Intersection Theorem, which in turn is known to imply Menger's Theorem, the Disjoint Spanning Tree Theorem and many other graph-theoretical results.

These are the main aspects of submodularity this paper will survey. We shall concentrate on some of the fundamental ideas and constructions; only a few proof will be described in detail, in cases when no appropriate reference can be given. This paper cannot undertake the task of a comprehensive survey of all aspects of submodularity, in particular if we consider matroid theory as a special case; for this we refer the interested reader to Welsh (1976), and shall assume that the reader is at least in part familiar with it. A forthcoming book of A. Schrijver is also strongly recommended for further reading on related subjects.