Applications of the FKG Inequality and Its Relatives

R. L. Graham
Bell Laboratories, Department of Discrete Mathematics, 600 Mountain Avenue, Murray Hill, New Jersey 07974, USA

Introduction

In 1971, C. M. Fortuin, P. W. Kasteleyn and J. Ginibre [FKG] published a remarkable inequality relating certain real functions defined on a finite distributive lattice. This inequality, now generally known as the FKG inequality, arose in connection with these authors' investigations into correlation properties of Ising ferromagnet spin systems and generalized earlier results of Griffiths [Gri] and Harris [Har] (who was studying percolation models). The FKG inequality in turn has stimulated further research in a number of directions, including a variety of interesting generalizations and applications, particularly to statistics, computer science and the theory of partially ordered sets. It turns out that special cases of the FKG inequality can be found in the literature of at least a half dozen different fields, and in some sense can be traced all the way back to work of Chebyshev.

In this paper, I will survey some of this history as well as the more recent extensions and applications. I will also discuss various open problems along the way which I hope will convince the reader that this exciting area is still fertile ground for further research.

Background

We begin with an old result of Chebyshev (see [HLP]) which asserts that if f and g are both increasing (or both decreasing) functions on [0, 1] then the average value of the product fg is at least as large as the product of the average values of f and g, where the average is taken with respect to some measure μ on [0, 1].

In symbols, this is just

$$\int_0^1 fg\,d\mu \geqslant \int_0^1 f\,d\mu \int_0^1 g\,d\mu \tag{1}$$

In the case that μ is a discrete measure we can restate (1) as follows: If $f(k)$ and $g(k)$ are both increasing (or both decreasing) and $\mu(k) \geqslant 0$ for $k = 1, 2, 3, \ldots$, then

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$$\frac{\sum_k f(k)g(k)\mu(k)}{\sum_k \mu(k)} \geqslant \frac{\sum_k f(k)\mu(k)}{\sum_k \mu(k)} \cdot \frac{\sum_k g(k)\mu(k)}{\sum_k \mu(k)}$$

i. e.,

$$\sum_k f(k)g(k)\mu(k) \sum_k \mu(k) \geqslant \sum_k f(k)\mu(k) \sum_k g(k)\mu(k). \tag{2}$$

The proofs of (1) and (2) follow immediately by expanding the inequality

$$\sum_{i,j} (f(i)-f(j))(g(i)-g(j))\mu(i)\mu(j) \geqslant 0.$$

Basically, the FKG inequality represents a way of extending (2) to the situation in which the underlying index set is only *partially* ordered, as opposed to the *totally* ordered index set of integers occurring in (2). The setting is as follows. Let $(\Gamma, <)$ be a finite distributive lattice, i. e., Γ is a finite set, partially ordered by $<$, for which the two functions $\wedge$ (meet or greatest lower bound) and $\vee$ (join or least upper bound) defined by:

$$x \wedge y := \max\{z \in \Gamma : z \leqslant x, z \leqslant y\},$$
$$x \vee y := \min\{z \in \Gamma : z \geqslant x, z \geqslant y\}$$

are well-defined and satisfy the distributive laws:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$
$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

for all $x, y, z \in \Gamma$.

It is well known that any such lattice can be realized as a sublattice of the lattice of all subsets of some finite set partially ordered by inclusion and with $x \wedge y = x \cap y$ and $x \vee y = x \cup y$.

We now suppose $\mu : \Gamma \to \mathbb{R}_0$, the nonnegative reals, satisfies

$$\mu(x \wedge y)\mu(x \vee y) \geqslant \mu(x)\mu(y) \quad \text{for all } x, y \in \Gamma. \tag{*}$$

For reasons we shall mention later, a function μ satisfying (*) is often called *log supermodular*. Finally, a function $f : \Gamma \to \mathbb{R}$ is called *increasing* if

$$x \leqslant y \Rightarrow f(x) \leqslant f(g) \quad \text{for } x, y \in \Gamma$$

(with *decreasing* defined similarly).

The FKG inequality states:
If f and g are both increasing (or both decreasing) real functions on a finite distributive lattice Γ and $\mu : \Gamma \to \mathbb{R}_0$ is log supermodular then

$$\sum_{x \in \Gamma} f(x)g(x)\mu(x) \sum_{x \in \Gamma} \mu(x) \geqslant \sum_{x \in \Gamma} f(x)\mu(x) \sum_{x \in \Gamma} g(x)\mu(x). \tag{3}$$

The original proof of (3) was somewhat complicated [FKG]. Several years after (3) appeared, Holley found the following beautiful generalization: