7. Accelerated Systems of Reference

The applications discussed in Chaps. 3 to 5 show that problems involving moving bodies are often preferably solved in the rest axes of the body. This policy requires a coordinate transformation from laboratory to rest axes. When the motion is a uniform translation the appropriate transformation is the Lorentz transformation. For other motions, e.g., a uniform rotation, a more general kind of transformation is needed. It is to a study of these transformations that the present chapter is devoted. The introduction of new coordinates has several important consequences, for example,

1) that time and length measurements must be reexamined carefully; and
2) that fundamental equations such as Maxwell's must be recast in a more general form.

The second problem is discussed in Chap. 9. In the present chapter we discuss the first, and give an elementary presentation of the formalism by which physical laws are cast in a particularly concise and elegant form, valid in all coordinate systems.

7.1 Coordinate Transformations

In an inertial system a point-event (also termed "world-point" or "space-time-point") is characterized by four numbers $x^\alpha$. We are free, however, to characterize this event by four new numbers $\tilde{x}^\alpha$ (a "marker system"), related to the former ones by purely mathematical relationships

$$
\begin{align*}
\tilde{x}^0 &= c t = x^0(x^0, x^1, x^2, x^3) \\
\tilde{x}^1 &= x^1(x^0, x^1, x^2, x^3) \\
\tilde{x}^2 &= x^2(x^0, x^1, x^2, x^3)
\end{align*}
$$

(7.1)
\[ x^3 = x^3(x^0, x^1, x^2, x^3) \]

The number \( t \) is the coordinate-time. It is obviously possible to replace the \( x^\alpha \) by new coordinates \( x^\beta \), given by

\[ x^\beta = x^\beta(x^\alpha) \quad (7.2) \]

and to perform a chain of transformations. A transformation in which \( x^0 \) is left untouched is termed a space transformation.

The relationship between coordinate increments can be written as

\[ dx^\alpha = \sum_{\beta=0}^{3} \frac{\partial x^\alpha}{\partial x^\beta} dx^\beta \quad (7.3) \]

More explicitly, we have

\[
\begin{pmatrix}
    dx^0 \\
    dx^1 \\
    dx^2 \\
    dx^3
\end{pmatrix} =
\begin{pmatrix}
    \frac{\partial x^0}{\partial x^0} & \frac{\partial x^0}{\partial x^1} & \frac{\partial x^0}{\partial x^2} & \frac{\partial x^0}{\partial x^3} \\
    \frac{\partial x^1}{\partial x^0} & \frac{\partial x^1}{\partial x^1} & \frac{\partial x^1}{\partial x^2} & \frac{\partial x^1}{\partial x^3} \\
    \frac{\partial x^2}{\partial x^0} & \frac{\partial x^2}{\partial x^1} & \frac{\partial x^2}{\partial x^2} & \frac{\partial x^2}{\partial x^3} \\
    \frac{\partial x^3}{\partial x^0} & \frac{\partial x^3}{\partial x^1} & \frac{\partial x^3}{\partial x^2} & \frac{\partial x^3}{\partial x^3}
\end{pmatrix} \begin{pmatrix}
    dx^0 \\
    dx^1 \\
    dx^2 \\
    dx^3
\end{pmatrix} \quad (7.4)
\]

coordinate transformation

matrix \( A_x \)

The \( x^\alpha \rightarrow x^\alpha \) transformation can be inverted to yield

\[ x^0 = cT = x^0(x^0, x^1, x^2, x^3) \]
\[ x^1 = x^1(x^0, x^1, x^2, x^3) \]
\[ x^2 = x^2(x^0, x^1, x^2, x^3) \]
\[ x^3 = x^3(x^0, x^1, x^2, x^3) \quad (7.5) \]

The relationship between the coordinate increments is now

\[ dx^\alpha = \sum_{\beta=0}^{3} \frac{\partial x^\alpha}{\partial x^\beta} dx^\beta \quad (7.6) \]