1.9 The Standard Numbering \( \varphi \) of \( \mathbb{P}(1) \)

In the previous chapters we have studied different kinds of machines, the computable functions and recursive and r.e. sets. The theorems we have proved are generally of a constructive nature themselves. We have shown e.g.: "for any \( f \in \mathbb{P}(2) \), \( \bar{w}(f) \in \mathbb{P}(1) \)" (Theorem 1.2.12), "for any stack machine there is a Turing machine computing the same function" (Theorem 1.6.7), "for any r.e. \( A \subseteq \mathbb{N} \) there is some recursive \( B \subseteq \mathbb{N}^2 \) such that \( A \) is the projection of \( B \)" (Theorem 1.8.4). The proofs of these and many other theorems, however, yield more information. We have for example shown: "there is an effective procedure which for any register machine \( M \) produces a register machine \( M' \) such that \( f_{M'} = \bar{u}(f_M) \)", or "there is an effective procedure which for any register machine \( M \) produces a register machine \( M' \) such that \( f_{M'} \) is a total function and \( \text{dom}(f_M) \) is the projection of \( f_{M'}^{-1}(0) \)."

In order to formulate these observations precisely one can introduce standard names for register machines (or the corresponding computable functions) which are appropriate words. Then the effective procedures can be represented by computable functions on the names. Since computability on words can be reduced to computability on numbers (Theorem 1.7.5), we shall use numbers as names which yields a formally simpler theory.

In this chapter we shall define a standard numbering \( \varphi : \mathbb{N} \rightarrow \mathbb{P}(1) \) of the unary partial recursive functions. We shall prove that this numbering \( \varphi \) satisfies two fundamental effectivity properties, the utm-theorem and the smn-theorem, by which it is characterized already up to equivalence. The numbering \( \varphi \) will be defined in three steps:

- A notation \( \text{tm} : \mathbb{W}(\alpha) \rightarrow \text{TM} \) of a suitable class of tape machines \( \text{TM} \) for the computable functions \( \mathbb{W}(\{1\}) \rightarrow \mathbb{W}(\{1\}) \) is introduced.
- A notation \( \varphi' : \mathbb{W}(\alpha) \rightarrow \mathbb{P}(1) \) is derived by: \( \varphi'(w) : = \text{the function from } \mathbb{P}(1) \text{ computed by the tape machine } \text{tm}(w) \).
- By a standard numbering \( \nu_{\text{TP}} \) of \( \text{dom}(\text{tm}) = \text{dom}(\varphi') \) the numbering \( \varphi \) is obtained:
  \[ \varphi := \varphi' \nu_{\text{TP}}. \]

The definition of the notation \( \text{tm} \) is the crucial one. By Theorem 1.7.5, the mapping \( f \mapsto i^{-1}f_i \) where \( i : \mathbb{N} \rightarrow \mathbb{W}(\{1\}) \) is a standard numbering maps the computable functions \( f : \mathbb{W}(\{1\}) \rightarrow \mathbb{W}(\{1\}) \) onto \( \mathbb{P}(1) \). By Lemma 1.5.3 for any computable function \( f : \mathbb{W}(\{1\}) \rightarrow \mathbb{W}(\{1\}) \) there is a tape machine \( M \) over \( \{(B,1),(1),B\} \) with \( f = f_M \). By Lemma 1.1.7 we may assume that the states of the flowchart of \( M \) are natural numbers. Furthermore we may assume that \( M \) has only one exit. We shall define a simple notation for all tape machines with one exit over \( \{(B,1),(1),B\} \) the states of which are natural numbers.
1 EXAMPLE
Consider a flowchart for a tape machine.

```
  2
  \---> 4 +
     \--->
  1   0  1
  \--|--|
      \-->
      1
```

Then the following word denotes this flowchart:

```
((e:1,1)(l^2:R,l^4)(l^4:1,l^2,e),1^2,1)
```

Thus, a state \( i \in \mathbb{N} \) is denoted by the word \( l^i \) of length \( i \), a statement \( i:R,j \) is denoted by the word \"(l^i:R,l^j)\" of length \( i+j+5 \), etc. The above program is a list of such statement words followed by the name of the initial state and by the name of the final state.

For formal convenience we shall admit that there are two or more statement words for each state in the list. In this case only the leftmost one is considered. Furthermore we admit that the list is incomplete. If necessary we implicitly add statement-words \((1^p:B,1^q_e,l^q_e)\) where \( q_e \) is the final state. (Thus we shift some complications from the syntax to the semantics.) The precise definition of the notation is lengthy but not difficult.

2 DEFINITION (notation of tape machines)

(1) Let \( \Omega := \{1|B|(|)|:|,|R|L\} \) be an alphabet with 8 elements. (The symbol \(|\) is used for metalinguistic separation since the comma is an element of \( \Omega \), i.e., \( , \in \Omega \).)

(2) Let \( TM \) be the set of the unary tape machines with one exit over \((B,1),(1,B)\) such that the states are natural numbers.

(3) Define \( TS \in W(\Omega) \) (the tape statements) and \( TP \in W(\Omega) \) (the tape programs) as follows.

\[
TS := \left\{ "(1^m:B,1^n)" \mid b \in \{B,1,R,L\}; m,n \in \mathbb{N} \right\}
\]

\[
\cup \left\{ "(1^m:B,1^k,1^n)" \mid b \in \{B,1\}; k,m,n \in \mathbb{N} \right\}
\]

\[
TP := \left\{ "(z_1\ldots z_k,1^i,1^j)" \mid i,j,k \in \mathbb{N}; i \neq j; z_1,\ldots,z_k \in TS \right\}
\]

(4) A mapping \( \text{tm}: W(\Omega) \longrightarrow TM \) is defined as follows.

\[
\text{dom}(\text{tm}) := TP,
\]

\[
\text{tm}("(w,1^i,1^j)") := M \in TM
\]