Walsh Series in Polar Coordinates

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Walsh functions $\Psi_0, \Psi_1, \ldots$ have many applications to information theory (see [3]), especially to problems of pattern recognition and image enhancement. The Walsh system shares many properties with other orthonormal systems but is distinguished by the fact that each $\Psi_j$ is locally constant, takes on only the values $\pm 1$, and the intervals of constancy shrink uniformly to points as $j \to \infty$.

For two-dimensional problems the hybrid double Walsh system

\[
\Psi_k(x)\Psi_j(y)
\]

is often used. This approach is not entirely satisfactory. First, the supports of the functions (1) fill the unit square. Thus for problems which are circular in nature (e.g., data from sonar, air traffic control, or large tropical depressions) there is much waste storing data from corners which are both unwanted and unneeded. Secondly, intervals of constancy of (1) do not shrink uniformly to points as $k+j \to \infty$. And third of all, ordering the double Walsh system is problematic and delicate. For example, when considering convergence of double Walsh-Fourier series should one use rectangular sums, circular sums, or sums based on some other polygonal region? The difficulty in answering this question is illustrated by how little is known about convergence by any of these methods. Moreover, when the answers are known they are not always what one wants. For example, there exist $f \in L^p$, $1 < p < 2$, whose double Walsh-Fourier series diverge a.e. when summed over any regular polygonal region [4].

We introduce polar Walsh functions on the unit disc $\Delta_0$ which eliminate these difficulties. First, set

\[ w_0 \equiv 1 \]

and

\[ w_1(r, \theta) = \begin{cases} +1 & 0 \leq \theta < \pi \\ -1 & \pi \leq \theta < 2\pi \end{cases}, \]

$0 \leq r < 1$. For each $j \in \{2^\ell, 2^{\ell+1}\}$, $\ell = 1, 2, \ldots$ define sets $\Delta^{(i)}_j$, $i = 1, 2$, by the following process. If $\ell = 2m$ is even, write $j$ uniquely as

\[ j = 2^m + p2^m + q \]
where $0 \leq p < 2$, $0 \leq q < 2$ and set
\[
\Delta^{(1)}_j = \left\{ (r, \theta) : \sqrt{\frac{p}{2m}} \leq r < \sqrt{\frac{p+1}{2m}}, \frac{q\pi}{2m-1} \leq \theta < \frac{(q+\frac{1}{2})\pi}{2m-1} \right\},
\]
\[
\Delta^{(2)}_j = \left\{ (r, \theta) : \sqrt{\frac{p}{2m}} \leq r < \sqrt{\frac{p+1}{2m}}, \frac{(q+\frac{1}{2})\pi}{2m-1} \leq \theta < \frac{(q+1)\pi}{2m-1} \right\}.
\]

If $\ell = 2m+1$ is odd, write $j$ uniquely as
\[
j = 2^{2m+1} + p2^{m+1} + q
\]
where $0 \leq p < 2^m$, $0 \leq q < 2^{m+1}$ and set
\[
\Delta^{(1)}_j = \left\{ (r, \theta) : \sqrt{\frac{p}{2^m}} \leq r < \sqrt{\frac{p+\frac{1}{2}}{2^m}}, \frac{q\pi}{2^m} \leq \theta < \frac{(q+\frac{1}{2})\pi}{2^m} \right\},
\]
\[
\Delta^{(2)}_j = \left\{ (r, \theta) : \sqrt{\frac{p+\frac{1}{2}}{2^m}} \leq r < \sqrt{\frac{p+1}{2^m}}, \frac{q\pi}{2^m} \leq \theta < \frac{(q+1)\pi}{2^m} \right\}.
\]

Then $\{ \Delta^{(1)}_j : i = 1, 2, j = 2^\ell, 2^\ell + 1, \ldots, 2^{\ell+1} - 1 \}$ forms a partition of the interior of $\Delta_0$ for each $\ell \geq 1$.

Set $\rho_0 \equiv w_1$. For $\ell = 1, 2, \ldots$. Define $\rho_\ell$ on the interior of $\Delta_0$ by
\[
\rho_\ell(r, \theta) = \begin{cases} +1 & (r, \theta) \in \Delta^{(1)}_j \\ -1 & (r, \theta) \in \Delta^{(2)}_j \end{cases}
\]
for $2^\ell \leq j < 2^{\ell+1}$. For each integer $k \geq 2$ define the polar Walsh function $w_k$ by
\[
w_k(r, \theta) = \rho_{\ell_1}(r, \theta) \cdots \rho_{\ell_\nu}(r, \theta)
\]
where $k$ is written uniquely as
\[
k = 2^{\ell_1} + \cdots + 2^{\ell_\nu}
\]
and $\ell_1 > \ell_2 > \cdots > \ell_\nu \geq 0$. Finally, extend each polar Walsh function to the closed unit disc by
\[
w_k(1, \theta) = \lim_{r \uparrow 1} w_k(r, \theta)
\]
$0 \leq \theta < 2\pi$, $k = 0, 1, \ldots$.

One can show that the polar Walsh system is complete and orthonormal on $L^2(\Delta_0)$ if two dimensional Lebesgué measure is normalized by $\pi$. Thus for each $f$ integrable on $\Delta_0$ define polar Walsh-Fourier coefficients by
\[
\hat{f}(j) = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 f(r, \theta) w_j(r, \theta) r dr d\theta,
\]
$j = 0, 1, \ldots$. Since the $w_j$'s are constant on the sets $\Delta^{(i)}_j$, $i = 1, 2$, and the area of these sets diminishes to zero as $j \to \infty$, it is not difficult to see that the Riemann-Lebesgué lemma holds for polar Walsh-Fourier coefficients, i.e., $\hat{f}(j) \to 0$ as $j \to \infty$. On the other hand, this property nearly characterizes polar Walsh-Fourier coefficients.