We shall describe a way to construct wavelets on an open set $\Omega$ of $\mathbb{R}^n$ (this construction is a joint work with Y. MEYER and can be found in (1) ; the reader should look there for precisions), then we shall give a more explicit description of the two following points that are important for possible applications: The asymptotic behavior (wavelets that are localized around very small cubes which are far from the boundary of $\Omega$ are numerically identical to the "corresponding" wavelet on $\mathbb{R}^n$) and the fast decomposition algorithms (which are of a similar kind as in $\mathbb{R}^n$ except that the storage of more filters is needed).

1 Construction and properties of wavelets

This construction is related to the multiresolution algorithms that were invented by S. MALLAT and Y. MEYER and are described in other papers of this book.

We shall construct an orthonormal basis of wavelets of $L^2(\Omega)$ that are $C^{2m}$ ($m \in \mathbb{N}$). For that we define a new form of multiresolution analysis as follows. Let $Q_{j,k}$ be the cube defined by $2^j x - k \in [0,1]$ and $V_j$ the subspace of $L^2(\Omega)$ composed with functions $C^{2m}$, such that $\text{supp } f \subset \Omega$ and $f$, restricted to $Q_{j,k}$ is a polynomial of degree $2m + 1$ at most in each variable. The $V_j$ are then an increasing sequence of closed subspaces of $L^2(\Omega)$ whose union is dense in $L^2(\Omega)$.

We shall at first construct an orthonormal basis of each $V_j$. We define the B-spline $\sigma$ by

$$\hat{\sigma}(\xi) = \left( \frac{\sin \xi_1}{\xi_1} \right)^{2m+2} \ldots \left( \frac{\sin \xi_n}{\xi_n} \right)^{2m+2}$$

A basis of $V_j$ is then obtained by taking the functions

$$\sigma_{j,k}(x) = 2^{n/2} \sigma(2^j x - k)$$

such that $\text{supp } \sigma_{j,k} \subset \Omega$. These functions form a Riesz basis of $V_j$, i.e.

$$\| \sum \alpha_{j,k} \sigma_{j,k} \|_{L^2} \sim (\sum |\alpha_{j,k}|^2)^{1/2}$$
The constants that appear in the equivalence being independant of $j$. The set of $n$-uples $k/2^j$ such that $\text{supp} \sigma_{j,k} \subset \tilde{\Omega}$ is caracterised by

\[(2) \quad d\left(\frac{k}{2^j}, \partial\Omega\right) \geq \frac{m + 1}{2^j}\]

(if $d(x, y) = \sup |x_i - y_i|$). A function of $V_j$ is determined by its values on $\Lambda_j$. More precisely, if $F \in V_j$

\[(3) \quad c_1 \|F\|_2 \leq \left(\sum_{\lambda \in \Lambda_j} 2^{-nj}|F(\lambda)|^2\right)^{1/2} \leq c_2 \|F\|_2\]

where $0 < c_1 \leq c_2 < +\infty$. We construct two new bases of $V_j$. The first one is orthonormal and thus obtained: Let $G$ be the operator defined over $V_j$ by

\[G(F) = \sum_{k/2^j \in \Lambda_j} <F, \sigma_{j,k}> \sigma_{j,k}\]

by (1) $G$ is positive definite. Let then $\phi_{j,k}$ be

\[\phi_{j,k} = G^{-1/2}(\sigma_{j,k}),\]

the $\phi_{j,k}$ are the required basis.

The second basis is composed with cardinal splines, i.e. functions $L_{j,k}$ of $V_j$ such that $L_{j,k}(l/2^j) = \delta_{k,l}$.

If $L_{j,k} = \sum \gamma(k, k')\sigma(2^j x - k')$

then the matrix:

\[(\gamma(k, k'))_{k,k' \in 2^j \Lambda_j}\]

is the inverse of the matrix

\[(\sigma(k - k'))_{k,k' \in 2^j \Lambda_j}\]

(The existence of this inverse is assured by (3)). Let $W_j$ be the orthogonal complementary of $V_j$ in $V_{j+1}$. We shall construct an orthonormal basis of $W_j$. Such a basis is obtained by projecting over $W_j$ the functions $L_{j+1,k}$ such that $\frac{k}{2^{j+1}} \in \Lambda_{j+1} \setminus \Lambda_j$ and orthonormalizing the basis thus obtained by the "Gram matrix device" we already used. We thus obtain the wavelets $\psi_{j+1,k}$ we were looking for. The union of these bases yields an orthonormal basis of $L^2(\Omega)$. If this construction was made with $\Omega = \mathbb{R}^n$, we would obtain the "usual" wavelets, i.e., in this case, there are functions $\phi$ and $\psi^{(i)}$ such that