Abstract

A review is presented of the anomaly problem in quantum field theory. The problems encountered in evaluating the $\pi^0 \rightarrow 2\gamma$ decay are described, as well as the point-splitting method for dealing with these problems and deriving the chiral anomaly. We next show how the Wess-Zumino consistency conditions allow the understanding of these results in the context of the cohomology of Lie algebras. Then some concepts from the theory of characteristic classes are discussed, including the Chern character, in order to derive the descent equations which are the basis for the algebraic calculation of anomalies. Finally, the relation of the anomaly problem to the question of Schwinger terms in current algebra is clarified.

1. Introduction

The relationship between physics and mathematics is intimate but not always straightforward. Some areas of physics have yielded to precise mathematical analysis. In particular, it has by now been widely recognized that classical gauge field theory can be usefully described in the language of modern differential geometry [1]. This description has produced various results which would hardly have been achieved otherwise, for example the complete classification of self-dual solutions of the Yang-Mills equations [2]. In contrast, quantum field theory in four spacetime dimensions has up to now resisted all attempts at a rigorous mathematical formulation.

It is thus of particular interest that a significant connection has been discovered between certain problems in quantum field theory and some deep results of modern algebraic topology. This is all the more remarkable in that we are here talking about problems of eminent physical importance. Besides a host of more speculative applications, the chiral anomaly was and remains the key to our understanding of the $\pi^0 \rightarrow 2\gamma$ decay process. It not only explains the existence of this decay mode, despite contrary expectations based on traditional
arguments of current algebra, it also yields a correct numerical value for the decay rate, upon inclusion of a factor of three due to the $SU(3)$-color degrees of freedom. Indeed, it was in this connection that Gell-Mann first introduced the $SU(3)$ gauge group [3], which is the basis for the standard theory of the strong interactions, quantum chromodynamics (QCD).

The field theoretic treatment of anomalies is rather intricate, being intimately related to questions of regularization and renormalization of the terms in the perturbation expansion. The mathematical approach avoids these dynamical details and focuses attention on the symmetry aspects of the problem. This is a strategy which has frequently been used to advantage in elementary particle physics. In contrast to the most common physical situations we have in this case a non-trivial realization of the symmetry, connected to a non-vanishing cohomology class, i.e. to a non-trivial topology. Phenomena related to non-trivial topological effects are relatively new in physics, and are often the occasion of some initial confusion. The Bohm-Aharonov effect in quantum mechanics [4] and Dirac's magnetic monopole [5] are cases in point. However, from the mathematician's point of view the treatment of such situations is completely straightforward. At this point it behooves the physicist to learn enough of the underlying mathematics to understand the resolution of the physical problem.

Besides the gain in understanding, the differential geometric approach has proved to be an efficient tool for calculating anomalies in rather general settings, where the perturbative calculations would be at best extremely unwieldy [6].

In this article we first review in some detail the problem of the chiral anomaly in quantum field theory and its relevance for the $\pi^0 \rightarrow 2\gamma$ decay. We emphasize in particular in what way the result of these calculations is unexpected from the point of view of a naive application of symmetry arguments in the framework of the canonical formalism. We illustrate one of the field-theoretic methods for overcoming these difficulties, involving Schwinger's gauge invariant point-splitting procedure. We then indicate how, by the use of the Wess-Zumino consistency conditions, the problem may be related to a question in the cohomology of Lie algebras. This requires using the differential geometric formulation of the gauge theory in terms of connections on a principal fibre bundle, and an understanding of the rôle the BRS transformations play in incorporating the restrictions due to gauge invariance and the accompanying unphysical degrees of freedom. Having at this stage reformulated the problem in a mathematical language, we go on to show how such problems are treated in the relevant mathematical framework. The concept of the Chern character, whose origin is in the theory of characteristic classes, is introduced. The basic transgression formula of Chern and Weil concerning the independence of the Chern class on the connection used in its construction is derived. An expansion of this formula according to degree yields the descent equations, which are the basic tool in this approach for the calculation of cohomology classes. The