1 Introduction

The concepts of stationarity and pseudo duality (P.D.) were introduced in PASSY and YOTAV (1980). It was shown how to calculate pairs of P.D. programs via Legendre transformation. Despite the fact that the class of functions having such transform is restricted, it was possible to generate P.D. to a large class of programs. The techniques that were used in PASSY and YOTAV (1980) could be applied again and it is assumed, without loss of generality, that all functions treated in the present paper have a Legendre transform. It will further be assumed that the reader is familiar with the notation and terminology used in PASSY and YOTAV (1980).

Remark 1. Functions that are related to each other through the Legendre transform are analyzed throughout the paper. Let \((\eta, \xi)\) be such a pair, then whenever such a pair is mentioned it is assumed, though not explicitly, that two subsets \((S, T)\) exist and \((h, S), (\xi, T)\) are a pair of Legendre conjugates, i.e.

\[
\begin{align*}
y &= \mathcal{V}h(x), & x \in S \Rightarrow y \in T \\
x &= \mathcal{V}\xi(y), & y \in T \Rightarrow x \in S.
\end{align*}
\]

Throughout the paper \(S\) denotes the domain of definition of the primal functions; objective and constraints, while \(T\) denotes the domain of definition of the dual functions.

The definitions of stationarity, except for the Lagrangian approach, were intuitive ones; a function is stationary if an infinitesimal feasible change in the argument causes a zero virtual change in the function value (cf. PASSY and YOTAV 1980). A direct extension of this approach is not possible for programs with inequality constraints as can be seen from example 1.

Example 1. Consider the following program:

\[
\text{Stat}\{f(x_1, x_2) | g(x) = x_1^2 + x_2^2 - 13 \leq 0\}.
\]

An intuitive definition of stationarity is the following: \(x^*\) is a solution of the above program if:

(i) \(\mathcal{V}f(x^*) = 0\) whenever \(g(x^*) < 0\)
(ii) \([\mathcal{V}f(x^*)]_\theta = 0\) for all \(\theta \ni \mathcal{V}g(x^*)\theta \leq 0\) if \(g(x^*) = 0\).

The point \(x^* = [3, 2]\) is feasible and is on the boundary, i.e., \(g(x^*) = 0\). The two vectors \(\theta_1 = [-4, 6], \theta_2 = [-5, 6]\) satisfy \(\mathcal{V}g(x^*)\theta_i \leq 0, i = 1, 2\) and are independent.
Thus

\[
\begin{align*}
Vf(x^*) \theta_1 &= 0 \\
Vf(x^*) \theta_2 &= 0
\end{align*}
\]

Hence \(x^*\) solves the program if it is a solution to the unconstrained program \(\text{Stat}\{f(x_1, x_2)\}\). Accordingly inequality constraints do not generate additional stationary points. This is not the case if constrained minimum and maximum are to be considered as stationary points.

Given a program

(1) \[ P: \quad \text{Stat}\{f(x) | x \in S \cap Q_1, g_i(x) \leq 0, j = 1, \ldots, l\} \]

where \(S\) is the domain of definiton of \(f\) and the \(g_i\)'s \(j = 1, \ldots, l\) and is a subset of \(\mathbb{R}^n\), \(Q_1\) is a \(k\)-dimensional subspace of \(\mathbb{R}^n\).

**Definition 1.** A point \(x^*\) solves \(P\) if:

(a) Feasibility: \(x^* \in S \cap Q_1, g_i(x^*) \leq 0 \quad j = 1, \ldots, l\)

(b) Stationarity: \(x^*\) solves the following program with equality constraints (see PASSY and YOTAV 1980)

\[ P(K): \quad \text{Stat}\{f(x) | x \in S \cap Q_1, g_i(x) = 0 \quad j \in K\} \]

for some \(K \subseteq K(x^*)\), where \(K(x^*) = \{j | g_j(x^*) = 0\}\).

An alternative definition of stationarity is obtained by transforming the inequality constraints into equality ones. Let \(\phi\) be a function defined on \(\mathbb{R}\) with the following properties: (i) \(\phi(s) \geq 0\); (ii) \(\phi(0) = 0\); (iii) \(\lim_{|s| \to \infty} \phi(s) = \infty\).

There are many functions that satisfy these conditions among which are \(s^2\) and \((\cosh(s) - 1)\).

**Definition 2.** A point \(x^*\) solves \(P\), equation (1), if \(\exists s^* \exists (x^*, s^*)\) solves the following program:

\[ \text{Stat}\{f(x) | x \in S \cap Q_1, s \in \mathbb{R}, g_i(x_j) + \phi_j(s_j) = 0 \quad j = 1, \ldots, l\} \]

The functions \(\phi_j\) satisfy conditions (i)–(iii) above.

If

(a) \(f, g_j \quad j = 1, \ldots, l\) and \(\phi_j \quad j = 1, \ldots, l\) are all differentiable

(b) \(\phi'_j(s_j) \neq 0 \quad s_j \neq 0\)

(c) at \(x^*\) the rows of the Jacobian matrix, \(J(g(x^*))\), generated from the tight constraints \(g_j(x^*) = 0\) are independent,

then the two definitions coincide and in this case the Kuhn-Tucker conditions are satisfied at \(x^*\), i.e.,

\[
\begin{align*}
V_x L(x^*, \lambda^*) &\in Q_1^\perp \quad (Q_1^\perp - \text{the orthogonal complement of } Q_1) \\
V_{\lambda} L(x^*, \lambda^*) &\leq 0 \\
\lambda_j g_j(x^*) &\leq 0 \quad j = 1, \ldots, l
\end{align*}
\]

where \(L\) is the Lagrangian function.