5. Structures in Banach Lattices

5.1 Banach Space Properties of Banach Lattices

In this section we mainly are interested in showing characterizations of properties of subspaces of Banach lattices. Moreover we will use the theory of order weakly compact operators to prove some results for arbitrary Banach spaces. First we will recall some basic facts concerning Schauder bases and topological embeddings of $c_0$.

Throughout this section let $X, Y$ be Banach spaces and $E, F$ Banach lattices.

**Definition 5.1.1.** i) A sequence $(x_n)_1^\infty \subset X$ is called a (Schauder) basis sequence if for every $x \in X$ contained in the closed linear span $Y$ generated by \{ $x_n : n \in \mathbb{N}$ \} there exists a unique sequence $(a_n)_1^\infty \subset \mathbb{R}$ satisfying

$$x = \sum_{n=1}^{\infty} a_n x_n.$$  

If $Y = X$, then the sequence $(x_n)_1^\infty$ is called a Schauder basis of $X$.

Moreover if $\|x_n\| = 1$, then $(x_n)_1^\infty$ is said to be a normalized basis sequence.

ii) Let $(x_n)_1^\infty$ be a basis sequence in $X$. A sequence of non-zero vectors $(y_n)_1^\infty \subset X$ of the form $y_n = a_{m_n+1}x_{m_n+1} + \ldots + a_{m_{n+1}}x_{m_{n+1}}$ with integers $m_1 = 0 < m_2 < \ldots < m_n < \ldots$ is called a block basis of the sequence $(x_n)_1^\infty$.

**Proposition 5.1.2.** A sequence $(x_n)_1^\infty \subset X$ is a basis sequence if and only if the following two conditions hold.

i) $x_n \neq 0$ for every $n \in \mathbb{N}$.

ii) There exists a so-called basis constant $K > 0$ satisfying

$$\left\| \sum_{i=1}^{n} a_i x_i \right\| \leq K \left\| \sum_{i=1}^{m} a_i x_i \right\|$$

for all $a_i \in \mathbb{R}$ and all $n, m \in \mathbb{N}$ with $n < m$.

**Proof.** First assume that $(x_n)_1^\infty$ is a basis sequence. By definition, i) holds.

ii) Let $Y$ be the closed linear span of \{ $x_n : n \in \mathbb{N}$ \}. For every $x \in Y$ let

$$q(x) = \sup \left\{ \left\| \sum_{i=1}^{n} a_i x_i \right\| : n \in \mathbb{N} \right\}$$

where $x = \sum_{n=1}^{\infty} a_n x_n$. 

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Obviously, \( q \) is finite and a norm on \( Y \) satisfying \( \|x\| \leq q(x) \) for every \( x \in Y \). A simple standard argument will show that \((Y, q)\) is complete. The open-mapping theorem asserts that there exists \( K > 0 \) such that \( K\|x\| \leq q(x) \) for all \( x \in Y \).

To prove the converse, assume that \( \sum_{n=1}^{\infty} a_n x_n = 0 \). Condition i) and ii) imply that \( a_n = 0 \) for all \( n \in \mathbb{N} \). Thus the representation is unique. Furthermore, condition ii) implies that the subspace consisting of all \( x = \sum_{n=1}^{\infty} a_n x_n \) is a closed linear subspace of \( X \).

**Proposition 5.1.3.** Let \((x_n)_1^\infty\) be a normalized basis sequence with the basis constant \( K > 0 \). If \((y_n)_1^\infty \subset X \) is a sequence satisfying

\[
\sum_{n=1}^{\infty} \|x_n - y_n\| < (2K)^{-1},
\]

then \((y_n)_1^\infty\) is a basis sequence equivalent to the sequence \((x_n)_1^\infty\).

Hereby two bases sequences \((x_n)_1^\infty\) and \((y_n)_1^\infty\) are called equivalent whenever

\[
\sum_{n=1}^{\infty} a_n x_n \text{ converges if and only if } \sum_{n=1}^{\infty} a_n x_n \text{ is convergent}.
\]

**Proof.** Let \( Z \) be the linear hull of \( \{x_n : n \in \mathbb{N}\} \) and \( Y \) the linear hull of \( \{y_n : n \in \mathbb{N}\} \). For every sequence \((a_n)_1^\infty \subset \mathbb{R} \) and \( m, p \in \mathbb{N} \) it follows that

\[
\left\| \sum_{n=m}^{m+p} a_n x_n - \sum_{n=m}^{m+p} a_n y_n \right\| \leq \sup \left\{ |a_n| \sum_{n=m}^{\infty} \|x_n - y_n\| : m \geq n \right\}.
\]

This shows that the sequences \((x_n)_1^\infty\) and \((y_n)_1^\infty\) are equivalent.

Now consider the mapping \( T : Z \to Y \) defined by

\[
T\left( \sum_{n=1}^{\infty} a_n x_n \right) = \sum_{n=1}^{\infty} a_n y_n.
\]

It follows that

\[
\|Tx - x\| \leq \sum_{n=1}^{\infty} |a_n| \|x_n - y_n\| \\
\leq \max \{|a_n| : n \in \mathbb{N}\} \sum_{n=1}^{\infty} \|x_n - y_n\| \leq K (2K)^{-1} = 1/2.
\]

Consequently the range of \( T \) is closed. This shows that \( T(Z) = Y \).

**Proposition 5.1.4.** Let \((x_n)_1^\infty\) be a Schauder basis of \( X \). If \((y_n)_1^\infty\) is a normalized sequence satisfying \( y_n \rightharpoonup 0 \) weakly as \( n \to \infty \), then there exists a subsequence \((y_{k(n)})_1^\infty\) equivalent to a block basis of \((x_n)_1^\infty\).