9. Unified Gauge Theories

9.1 Introduction: The Symmetry Group SU(5)

The Glashow–Salam–Weinberg theory combines the electromagnetic and weak interactions within the framework of the gauge group SU(2)\(_L\) × SU(1). This theory treats leptons and quarks on the same footing. However, the fact that leptons carry integer charge while the charge of quarks is 1/3 is needed as basic input. On the other hand, there are indications that this fact is due to some superior principle, that is a larger symmetry. For example, for leptons as well as for quarks, the sum of charge and baryon number, \(Q + B\), is an integer:

\[
\begin{align*}
\text{e: } & Q = -1, B = 0 \quad \rightarrow \quad Q + B = -1, \\
\nu: & Q = 0, B = 0 \quad \rightarrow \quad Q + B = 0, \\
\text{u: } & Q = +\frac{2}{3}, B = \frac{1}{3} \quad \rightarrow \quad Q + B = 1, \\
\text{d: } & Q = -\frac{1}{3}, B = \frac{2}{3} \quad \rightarrow \quad Q + B = 0.
\end{align*}
\]

The property that \(B\) and \(Q\) are multiples of 1/3 for quarks is connected to the principle that baryons consist of three quarks and have integer values of \(Q\) and \(B\). Within the framework of quantum chromodynamics (QCD) this is understood in terms of the gauge group SU(3) of additional internal degrees of freedom for the quarks, the colour. For example, in order to explain the \(\Omega^-\) baryon it requires the assumption of three strange quarks all in the same \(1s\)-state. This is inconsistent with the Pauli principle, which is assumed as a fundamental property, unless the three s-quarks differ in an additional quantum number. These quantum states are labeled by colour and are distinguished as, for example red \(r\), green \(g\) and blue \(b\).

Similar relations to those above can also be stated in terms of the weak hypercharge. The lepton doublets have \(Y_L^{(e)} = -1\), while for quarks \(Y_L^{(q)} = +\frac{1}{3}\). Owing to the colour multiplicity the latter needs to be counted three times, leading to

\[
Y_L^{(q)} + 3 Y_L^{(q)} = 0.
\]

Similar relations can be inferred for the right-handed singlets.

These properties lead to the hypothesis that the gauge group of the Glashow–Salam–Weinberg theory, SU(2)\(_L\) × U(1), and the SU(3) group of the strong interaction form part of a larger symmetry. The simplest group that incorporates the product SU(3) × SU(2) × U(1) as a subgroup is SU(5). As will be shown below, this group has irreducible representations of dimension

\[5, \ 10, \ 15, \ 24, \ 35, \ 40, \ \ldots\]
Before discussing the details of this model of unification of the three interactions, which was proposed by Georgi and Glashow in 1974, it is expedient to establish some basic mathematical properties of the group SU(5).

SU(5) is a special case of the general groups SU(n) that are formed from unitary \( n \times n \) matrices with determinant (+1). The corresponding groups without the constraint for the determinant are called U(n). An arbitrary unitary matrix can be represented in terms of an exponential of a Hermitian matrix \( \hat{H} \):

\[
\hat{U} = \exp(i\hat{H}) , \quad \hat{H}^\dagger = \hat{H} .
\]

(9.1)

\( \hat{H} \) is called the \textit{generating matrix} for \( \hat{U} \). In case that the matrix \( \hat{U} \) is not too different from the unit matrix \( \mathbb{I} \), it holds

\[
\hat{U} = \exp(i\delta \hat{H}) \approx \mathbb{I} + i\delta \hat{H} .
\]

(9.2)

The multiplication of two matrices \( U_1, U_2 \) corresponds to the sum of the infinitesimal Hermitian matrices,

\[
\hat{U}_2 \hat{U}_1 = \exp(i\delta \hat{H}_2) \exp(i\delta \hat{H}_1) \approx (\mathbb{I} + i\delta \hat{H}_2) (\mathbb{I} + i\delta \hat{H}_1) \\
\approx \mathbb{I} + i(\delta \hat{H}_2 + \delta \hat{H}_1) ,
\]

(9.3)

where quadratic terms have been neglected. A complete set of linearly independent, Hermitian matrices is termed a set of generators for the unitary matrices. Owing to the constraint \( h_{ik} = h_{ki}^* \) for the elements of a Hermitian matrix \( \hat{H} \), the group of the unitary \( n \times n \) matrices contains \( n^2 \) generators. (Note that a general complex \( n \times n \) matrix has \( 2n^2 \) degrees of freedom.) The restriction \( \det(U) = 1 \) that leads from the group U(\( n \)) to the group SU(\( n \)) results in traceless generating matrices \( \hat{H} \). Since the diagonal elements of Hermitian matrices are real, \( h_{kk} = h_{kk}^* \), only one degree of freedom is omitted, That is the group SU(\( n \)) has \( n^2 - 1 \) generators.

The simplest representation of the generating matrices for the U(\( n \)) is formed by certain \( n \times n \) matrices that contain a single nonvanishing matrix element with value 1, all other elements being zero. Such a matrix, where the matrix element with value 1 is given by the intersection of row \( \alpha \) and column \( \beta \) as \( \hat{C}_{\alpha \beta} \), is written

\[
\hat{C}_{\alpha \beta} = \alpha \left( \begin{array}{ccc} -- & 1 & -- \\ \end{array} \right) , \quad (C_{\alpha \beta})_{ik} = \delta_{\alpha i} \delta_{\beta k}
\]

(9.4)

where the matrices

\[
\hat{C}_{\alpha \beta} + \hat{C}_{\beta \alpha} , \quad \frac{1}{i} (\hat{C}_{\alpha \beta} - \hat{C}_{\beta \alpha})
\]

(9.5)

are Hermitian, since \( \hat{C}_{\alpha \beta} = \hat{C}_{\beta \alpha}^\dagger \).

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