VI Nonintegrability in the Vicinity of an Equilibrium Position

There is another method of proving nonintegrability. It is based on estimates from below for coefficients of the power series representing formal integrals which exist by Birkhoff's theorem. It turns out that the reason for divergence of such series is the existence of anomalously small denominators, i.e., almost resonance relations between the frequencies of small oscillations near equilibria.

1 Siegel's Method

1.1 Consider a canonical system of differential equations

\[ \dot{x}_k = \frac{\partial H}{\partial y_k}, \quad \dot{y}_k = -\frac{\partial H}{\partial x_k}, \quad 1 \leq k \leq n, \quad (1.1) \]

and assume that \( H \) is an analytic function in a neighborhood of the point \( x = y = 0 \), where \( H(0) = 0 \) and \( dH(0) = 0 \). Let \( H = \sum_{s \geq 2} H_s \), where \( H_s \) is a homogeneous polynomial in \( x \) and \( y \) of degree \( s \).

Let \( \lambda_1, \ldots, \lambda_n \) be the eigenvalues of the linearized canonical system with Hamiltonian \( H_2 \). We may assume that \( \lambda_{n+k} = -\lambda_k \) (\( 1 \leq k \leq n \)). We consider the case when the numbers \( \lambda_1, \ldots, \lambda_n \) are purely imaginary and independent over the field of rational numbers: the sum \( m_1 \lambda_1 + \cdots + m_n \lambda_n \) with integer \( m_i \) equals zero only if all \( m_i = 0 \). Under this assumption, Birkhoff found a formal canonical transformation reducing the system \( (1.1) \) to the normal form. In particular, the Hamiltonian equations \( (1.1) \) have \( n \) integrals in involution in the form of formal series in \( x \) and \( y \) (Chap. II, §11).

In this section we investigate obstructions to the complete integrability of the system \( (1.1) \) in the vicinity of the equilibrium position \( x = y = 0 \) and the convergence of Birkhoff's normalizing transformations. Let \( \mathcal{H} \) be the set of all power series

\[ H = \sum h_{ks} x^k y^s, \quad k = (k_1, \ldots, k_n), \quad s = (s_1, \ldots, s_n), \]

converging in some neighborhood of the point \( x = y = 0 \). We endow \( \mathcal{H} \) with the following topology \( \mathcal{T} \): a neighborhood of a power series \( H^* \) with coefficients \( h_{ks}^* \) is the set of all power series \( H \) whose coefficients \( h_{ks} \) satisfy the inequalities \( |h_{ks} - h_{ks}^*| < \varepsilon_{ks} \), where \( \varepsilon_{ks} \) is an arbitrary sequence of positive numbers.

Theorem 1 (Siegel [215]). In every neighborhood of an arbitrary point \( H^* \in \mathcal{H} \) there exists a Hamiltonian \( H \) such that the corresponding canonical system \( (1.1) \)
Siegel's Method does not have integrals independent of $H$ and analytic in a neighborhood of the point $x = y = 0$.

The proof is contained in §1.2. The theorem implies that nonintegrable systems are everywhere dense in $\mathcal{H}$. In particular, the set of Hamiltonian systems for which the Birkhoff transformation diverges is everywhere dense. Concerning the divergence of Birkhoff's transformations, the following stronger result holds:

**Theorem 2** (Siegel [216]). *The Hamiltonian functions $H \in \mathcal{H}$ with converging Birkhoff transformations form a subset of first Baire category in the topology $\mathcal{T}$."

Recall that a subset of a topological space is a set of first Baire category if it is a countable sum of nowhere dense sets. A set is nowhere dense if every neighborhood of any point of the topological space contains a non-empty open domain with no points from this set.

More precisely, Siegel proved the existence of a countably infinite set of independent analytic power series $\Phi_1, \Phi_2, \ldots$ in an infinite number of variables $h_{ks}$ which are absolutely convergent for $|h_{ks}| < \varepsilon$ (for all $k, s$), and such that if $H \in \mathcal{H}$ is reduced to the normal form by a convergent transformation, then almost all $\Phi_s$ (except possibly a finite number) vanish at the point $H$. Since the functions $\Phi_s$ are analytic, their zeros are nowhere dense in $\mathcal{H}$. Consequently, the set of points of $\mathcal{H}$ satisfying at least one equation $\Phi_s = 0$ is of first Baire category. If we attempt to investigate the convergence of the Birkhoff transformation for any concrete Hamiltonian system, then we must check infinitely many conditions. No finite method for doing this is known, although all the coefficients of the series $\Phi_s$ can be calculated explicitly.

For example, it is still unknown whether the Birkhoff transformation is convergent near the Lagrangian positions of equilibrium in the restricted three-body problem with a fixed mass ratio. Concerning this problem, Siegel remarked that "... probably it is beyond the possibilities for known methods of calculus" [216].

1.2 According to the results of §11, Chap. II, in certain canonical coordinates the Hamiltonian $H$ takes the form:

$$H = \frac{1}{2} \sum \omega_j (x_j^2 + y_j^2) + \cdots.$$  

The eigenvalues are precisely $\pm i \omega_1, \ldots, \pm i \omega_n$. Perform the following linear canonical transformation $x, y \to u, v$ with complex coefficients:

$$y = \frac{iu + v}{\sqrt{2}}, \quad x = \frac{u + iv}{\sqrt{2}}.$$  

In the new coordinates

$$H = i \sum \omega_j u_j v_j + \cdots.$$  

For the sake of simplicity, we restrict ourselves to the case of two degrees of freedom. Let $\omega_1 = 1$ and $\omega_2 = \omega$ be irrational.