6 The Planetary Disturbing Function

6.1 General Structure

The main difficulty in expanding the disturbing function in planetary problems is that the ratio of the semi-major axes of the planetary orbits may be rather large (of the order of 0.7, for example, for the pair Venus–Earth). Therefore, efficient expansions in powers of this ratio in satellite problems cannot be directly used here. They are given below to characterize the general structure of the expansions.

Since the expansion of the indirect term of the disturbing function causes no difficulties we consider here the reciprocal of the mutual distance $\Delta$ between the planets. Some methods of planetary perturbation theory demand the expansion of not only $\Delta^{-1}$ but $\Delta^{-3}$, $\Delta^{-5}$, ... as well. Taking this into account we shall treat the problem of expanding $\Delta^{-n}$ for an arbitrary natural integer $n$.

In dealing with the function

$$\Delta = (r^2 + r'^2 - 2rr' \cos H)^{\frac{1}{2}}$$

(6.1.1)

we shall choose the case $r < r'$. Then the standard expansion in Gegenbauer polynomials results in

$$\Delta^{-n} = r'^{-n} \sum_{k=0}^{\infty} \left( \frac{r}{r'} \right)^k C_k^{n/2}(\cos H).$$

(6.1.2)

The expansion of the Gegenbauer polynomials in terms of true anomalies $v$, $v'$, arguments of pericentres $\omega$, $\omega'$, longitudes of nodes $\Omega$, $\Omega'$ and inclinations $i$, $i'$ may be expressed in the form

$$C_k^{n/2}(\cos H) = \sum_{l=0}^{k} \sum_{l'=0}^{k} \sum_{j=0}^{k} (2 - \delta_{j0})C_{k-l-l',l'-l,j}^{(k,n/2)}(i,i') \times \cos[(k - 2l)(v + \omega) - (k - 2l')(v' + \omega') + j(\Omega - \Omega')].$$

(6.1.3)

The coefficients $C_{pq}^{(k,n/2)}(i,i')$ admit a simple representation in the case of Legendre polynomials for $n = 1$: 

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\[ C^{(k,1/2)}_{pqj}(i, i') = \frac{(1)^{k-j}}{(1)^{k+j}} F_{k,j, \frac{k-2}{2} - q}(i) F_{k,j, \frac{k-2}{2} + q}(i'), \]  

(6.1.4)

\( F_{kjl}(i) \) being the Kaula inclination function (5.3.10). The representation of the Gegenbauer polynomial as the linear combination of a set of Legendre polynomials (Sack, 1964)

\[ C_k^{n/2}(x) = \sum_{r=0}^{\infty} \frac{E\left(\frac{n}{2}\right)}{\left(2k + 4r + 1\right)} \frac{\left(\frac{n}{2}\right)_{k+r}}{\left(\frac{3}{2}\right)_{k-r}} C_{k-2r}^{1/2}(x) \]

enables us to express the coefficients of expansion (6.1.3) as the sum of the products of the Kaula inclination functions:

\[ C^{(k,n/2)}_{pqj}(i, i') = \sum_{r=0}^{\infty} \frac{E\left(\frac{n}{2}\right)}{\left(2k - 4r + 1\right)} \frac{\left(\frac{n}{2}\right)_{k-r}}{\left(\frac{3}{2}\right)_{k-r}} C_{k-2r}^{1/2}(x) \]

\[ \times \frac{(1)^{k-2r-j}}{(1)^{k-2r+j}} F_{k-2r,j, \frac{k-2r+j-2l}{2} - q}(i) F_{k-2r,j, \frac{k-2r+j+2l}{2} + q}(i'). \]

(6.1.6)

The summation is performed here up to the first vanishing term. Therefore, the limiting value \( r = r^* \) is determined by

\[ r^* = \min \left\{ E\left(\frac{k - j}{2}\right), \frac{k - \max|p| - \max|q|}{2} \right\}. \]

(6.1.7)

From (6.1.3) it is evident that \( p + q \) and \( k \) in (6.1.4), (6.1.6) and (6.1.7) are integers of the same parity. Formula (6.1.6) may be transformed to reveal the dependence on \( i \) and \( i' \) more explicitly. Indeed, in accordance with (5.3.9) and (5.3.10) one may put (Brumberg, 1967)

\[ F_{kjl}(i) = \lambda_{kjl} \left( \sin \frac{i}{2} \right)^{|k-j-2l|} \left( \cos \frac{i}{2} \right)^{|k+j-2l|} \]

\[ \times F\left( \frac{1}{2} |k-j-2l| + \frac{1}{2} |k+j-2l| - k, \right. \]

\[ \left. 1 + k + \frac{1}{2} |k-j-2l| + \frac{1}{2} |k+j-2l|, \right. \]

\[ 1 + |k-j-2l|, \sin^2 \frac{i}{2} \]

(6.1.8)

with

\[ \lambda_{kjl} = (-1)^{E\left(\frac{k-j}{2}\right)} \max\{0, -k+j+2l\} \frac{(1)^{k+j}}{2^k (1)_{1}(1)_{k-l}} \times \]

\[ \times \frac{(1 + k + j)\max\{0, k-j-2l\} \max\{0, -k+j+2l\}}{(1)_{|k-j-2l|}} \]

(6.1.9)

Substituting (6.1.8) into (6.1.6) one gets