Error estimation and Adaptivity: 
Achievements of the last decade

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1. INTRODUCTION

The approximation involved in discretizing continuum problems by the finite element method may result in unacceptable errors. Much effort has been directed to keeping such errors under control so that engineering decision can be confidently made. Indeed, ideally the user will seek a solution with an a-priori specific accuracy tolerance to avoid excessive cost of providing an over-accurate solution.

The process of such error control is known as adaptivity and involves three essential stages:

1. The determination of error on a given mesh
2. The prediction of refinement of the mesh necessary to achieve specified accuracy
3. The generation of the required refinement.

In general the most obvious refinement consists of the use of the same elements and change of element sizes by mesh subdivision (or preferably regeneration) - this is the so called h-refinement. An alternative is to keep the element sizes unchanged and to vary the order of the shape function polynomials - this is the well known p-refinement.

Of course the most economical route of obtaining the satisfactory solution should be followed - and the so-called h-p refinement is an attempt to optimize the possibilities.

All three stages of the adaptive process are important and subject to much research. However the first, that of error estimation, is crucial and we shall discuss it hence in more detail.

2. ERROR ESTIMATION

There appear to be two distinct forms of error estimators:

(I) Estimators based on evaluation of residuals and an auxiliary local solution of the original problem.

and

(ii) Estimators based on comparison of the direct finite element solution with a more accurate recovered solution obtained by post-processing.

The first form of error estimator was introduced by Babuska and Rheinboldt [1,2] and is the basis of much further work by others. In particular the procedures used in this error estimator have been extended to quadratic and higher order elements [3] and improved in quality by ensuring the equilibrium of residual forces used in the local solutions [4-8].
The second method of error estimation was introduced in 1987 by Zienkiewicz and Zhu [9]. In this simple stress averaging or $L^2$ recovery were first used with remarkable success. Though this second method is intuitively obvious, simple in principle and extremely effective, some opposition was initially put up by mathematicians [10] as its derivation is too simple - and much dependent on the recovery process used.

In 1992 the authors put forward a new recovery process - based on the existence of superconvergent (or at least improved) sampling points within elements and a process of polynomial smoothing over an element patch [11-14]. This is now called the superconvergent recovery process - SPR.

With this improved recovery the error estimation performance improved to the extent that Babuska stated [15], after performing a wide ranging comparative study

that for “computing the existing estimators based on residuals and smoothing (i.e. recovery) the new ZZ estimator is the most robust and should be preferred in general”.

The same conclusions are reached in ref. [16,17] though the statement there is a little confusing and imprecise.

The recovery process used in the error estimators of the second kind is generally confined to the derivatives of the solution (e.g. in displacement analysis to stresses or strains) and is generally applicable to self adjoint problems. However unlike the first kind of estimators - (based on residuals) it allows any norm of error to be used in the estimate.

Indeed it can be easily shown [12] that the effectivity of the error estimates which is defined by

$$\theta = \frac{\text{estimated error}}{\text{actual error}}$$

(1)

is bounded by

$$1 - \frac{\|e\|}{\|e^*\|} \leq \theta \leq 1 + \frac{\|e\|}{\|e^*\|}$$

(2)

where $\|e\|$ is any norm of the actual error and $\|e^*\|$ is the same norm of the recovered solution error.

The obvious corollary of above theorem is the fact that an asymptotically exact error estimate is achieved if the recovered stresses converge at a higher rate than the original solution.

The interest in the patch recovery procedures introduced by us appear to be growing rapidly with many “additional improvements” used.

The work of Wiberg et al. [18,19] and Belytschko et al [20,21] introduces some of these and we propose to discuss these in the presentation.