12. Relation Between Birth Rates and Death Rates

Alfred J. Lotka (1907)

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A short notice appeared on page 641 of Science, 1907, of a paper read by C. E. Woodruff before the American Association for the Advancement of Science, on the relation between birth rates and death rates, etc.

In this connection, it may be of interest to note that a mathematical expression can be obtained for the relation between the birth rate per head $b$ and the death rate per head $d$, for the case where the general conditions in the community are constant, and the influence of emigration and immigration is negligible.

Comparison with some figures taken from actual observation shows that these at times approach very nearly the relation deduced on the assumptions indicated above.

I give here the development of the formula, and some figures obtained by calculation by its aid, together with the observed values, for comparison.

Let $c(a)$ be such a coefficient that out of the total number $N_t$ of individuals in the community at time $t$, the number whose age lies between the values $a$ and $(a + da)$ is given by $N_t c(a) da$.

Now the $N_t c(a) da$ individuals whose age at time $t$ lies between the values $a$ and $(a + da)$, are the survivors of the individuals born in time $da$ at time $(t - a)$.

If we denote by $B_{(t-a)}$ the total birth rate at time $(t - a)$, and by $p(a)$ the probability at its birth, that any individual will reach age $a$, then the number of the above-mentioned survivors is evidently $B_{(t-a)} p(a) da$.

Hence:

$$N_t c(a) da = B_{(t-a)} p(a) da$$

$$c(a) = \frac{B_{(t-a)} p(a)}{N_t}$$

Now if general conditions in the community are constant, $c(a)$ will tend to assume a fixed form. A little reflection shows that then both $N$ and $B$ will increase in geometric progression with time, at the same rate $r=(b-d)$. We may, therefore, write:

$$B_{(t-a)} = B_t e^{-ra}$$

$$c(a) = \frac{B_t}{N_t} e^{-ra} p(a)$$

$$= b e^{-ra} p(a)$$ (1)

Now from the nature of the coefficient $c(a)$ it follows that

$$\int_{0}^{\infty} c(a) da = 1$$

Substituting this in (1) we have:

$$\frac{1}{b} = \int_{0}^{\infty} e^{-ra} p(a) da$$ (2)

\[1\] Compare M. Block, "Traité théorique et pratique de statistique," 1886, p. 209.