3. Medium of Complex Structure

A movement of a particle in a medium of complex structure is described by the displacement of the mass center of the particle and by additional internal degrees of freedom: microrotations and microdeformations. In this chapter starting from simple models, we develop a general theory of media of complex structure. Essential attention is paid to new physical effects, approximate models and a transition to the classical local models.

3.1 Basic Micromodels

The diatomic chain or the corresponding mechanical model with masses and elastic bonds (Fig. 3.1) can serve as a simplest model of medium of complex structure.

It is assumed that the chain is separated into elementary cells, each consisting of two masses, which are connected by elastic bonds with a certain number of neighbors (the interaction of nearest neighbors is shown symbolically in the diagram). Let us first consider a homogenous diatomic chain in which, the structure and characteristics of each cell are the same.

Let \( n \) be the number of a cell, \( m_j (j = 1, 2) \) be the masses of the particles in the cell, \( w(n, j) \) be the displacement of the \( j \)-th particle in the \( n \)-th cell, \( f(n, j) \) be an external force acting on the particle. Then the equations of motion of particles in the \( n \)-th cell have the form

\[
m_j \ddot{w}(n, j) + \sum_{\delta \rho} \Phi(n - n', j, j') w(n', j') = f(n, j),
\]

where \( \Phi(n - n', j, j') \) are force constants, which are determined by the properties of the elastic bonds. Their physical meaning follows directly from the equations: if a displacement of the unit value of the particle \( (n', j') \) is the only one which differs from zero, then an external force \( \Phi(n - n', j, j') \), which compensates the reaction of elastic bond, must be applied to the particle \( (n, j) \). The dependence of the force constants on the difference \( n - n' \) is the consequence of the identical structures of the elementary cells. It is assumed that action at a distance is bounded, i.e. \( \Phi(n, j, j') = 0 \) when \( n > N \).

I. A. Kunin, Elastic Media with Microstructure I
© Springer-Verlag Berlin Heidelberg 1982
The equations of motion are the Euler's equations for the Lagrangian

\[
L \overset{\text{def}}{=} \frac{1}{2} \sum_{n,j} m_j \dot{w}^2(n,j) - \frac{1}{2} \sum_{m',j'} w(n,j) \Phi(n - n',j,j') w(n',j') + \sum_{n,j} f(n,j) w(n,j),
\]

wherefrom it follows that the force constants must satisfy the symmetry condition:

\[
\Phi(n - n',j,j') = \Phi(n',n,j',j).
\]

Finally, the force constants must satisfy additional conditions, due to the necessity of invariance of energy with respect to translation, i.e. with respect to transformation

\[
w(n,j) \rightarrow w(n,j) + w_0,
\]

where \(w_0\) is an arbitrary constant. It can be shown that this is equivalent to the condition of invariance of the equations of motion with respect to translation. Substituting (3.1.4) in (3.1.1), we find

\[
\sum_{n',j'} \Phi(n - n',j,j') = 0.
\]

These relations may be also interpreted as the definition of the self-action constants [compare with (2.1.11)]

\[
\psi(j) \overset{\text{def}}{=} \Phi(0,j,j) = - \sum_{n,j' \neq j} \Phi(n,j,j').
\]

**Problem 3.1.1.** Show that (3.1.5) implies the representation

\[
\Phi(n - n',j,j') = \psi(j) \delta(n - n') \delta(j - j') - \Psi(n - n',j,j'),
\]

where the \(\Psi(n - n',j,j)\) have the meaning of stiffness of elastic bonds and may be given arbitrarily (except the value \(\Psi(0,j,j) = 0\)).

The physical sense of the model of the medium of complex structure becomes more clear, if one passes to new variables: the displacement \(u(n)\) of the center of mass of the \(n\) - th cell and the relative displacement \(\eta(n)\) of particles in the cell, defined by the relations

\[
\begin{align*}
u(n) & \overset{\text{def}}{=} \frac{1}{m} [m_1 w(n,1) + m_2 w(n,2)], \\
\eta(n) & \overset{\text{def}}{=} I_*^{-1} [m_1 \xi_1 w(n,1) + m_2 \xi_2 w(n,2)], \\
m & \overset{\text{def}}{=} m_1 + m_2, \quad I_* \overset{\text{def}}{=} m_1 \xi_1^2 + m_2 \xi_2^2.
\end{align*}
\]