5. The Klein Paradox

The Klein paradox is connected with (but not identical to) the phenomena of vacuum change in supercritical fields. Since the latter are one of the major objectives of this book, we shall discuss the Klein paradox extensively. It is named after O. Klein, who in 1929 studied the scattering of electrons from a potential barrier of height $V_0$ [Kl 29]. We first study this paradox purely from the single-particle standpoint to interpret the Dirac equation, i.e. without use of hole theory, as Klein did. Later we introduce the hole-theory interpretation, as was originally done by Hund [Hu 54].

5.1 The Klein Paradox in the Single-Particle Interpretation of the Dirac Equation

Consider a case when electrons have energy $E$ and momentum $p_z$, a potential step appears at $z = 0$ and the potential is infinitely extended along the positive $z$ axis (Fig. 5.1).

For free electrons

$$\left(\frac{E}{c}\right)^2 = p^2 + m_0^2 c^2,$$  \hspace{1cm} (5.1a)$$

while for electrons in the constant potential

$$\left(\frac{E - V_0}{c}\right)^2 = \tilde{p}^2 + m_0^2 c^2$$  \hspace{1cm} (5.1b)$$

holds. The Dirac equation and its adjoint equation read

![Fig. 5.1. Potential set-up for the Klein paradox](image-url)
5.1 The Klein Paradox in the Single-Particle Interpretation of the Dirac Equation

The potential barrier lies at \( z = 0 \); hence we set

\[
e V = V_0 \quad \text{for} \quad z \geq 0,
\]
\[
e V = 0 \quad \text{for} \quad z < 0,
\]

and for the incoming plane wave obtain

\[
\psi_i = u_i \exp \left[ -i (Et - pz)/h \right],
\]

where \( u_i \) obeys (\( \hat{\alpha} = \hat{\alpha}_3 \))

\[
\left[ \frac{E}{c} - \hat{\alpha}p - \hat{\beta}m_0c \right] u_i = 0.
\]

Since we require \( u_i \neq 0 \), we conclude because of \( \hat{\alpha} \hat{\beta} + \hat{\beta} \hat{\alpha} = 0 \) that

\[
E^2/c^2 = p^2 + m_0^2c^2,
\]

which is (5.1a). We chose \( E > 0 \), because we are interested in an incoming electron. The reflected wave has to have momentum \( -\bar{p} \), while the momentum \( \bar{p} \) of the penetrating wave is given by (5.1b). For small \( V_0 \), \( \bar{p} \) is real and positive, so that

\[
\psi_r = u_r \exp \left[ -i (Et + pz)/h \right], \quad \psi_p = u_p \exp \left[ -i (Et - \bar{p}z)/h \right],
\]

where according to (5.2) \( u_r \) and \( u_p \) obey

\[
\left[ \frac{E}{c} + \hat{\alpha}p - m_0c \hat{\beta} \right] u_r = 0 \quad \text{and}
\]
\[
\left[ \frac{E - V_0}{c} - \hat{\alpha} \bar{p} - m_0c \hat{\beta} \right] u_p = 0
\]

respectively. The total wave function has to be smooth at the barrier (\( z = 0 \)):

\[
u_i + u_r = u_p.
\]

From (5.4, 7) it follows that

\[
\left[ \frac{E}{c} - m_0c \hat{\beta} \right] (u_i + u_r) = + \hat{\alpha}p(u_i - u_r),
\]

and from (5.7, 8)