CHAPTER 1

Kinematics of Flow

1.1 Introduction

This chapter examines some aspects of kinematic measures of motion. The discussion is limited to the classical model, in which microstructure is not considered, and where the motion is specified by the history of the particle velocity, so that there is no motion if the velocity field is identically zero. Theories in which there is motion even in the absence of a velocity distribution are considered in later chapters.

In the usual discussion on the kinematics of flow, an emphasis is placed on the rates of strain of material line elements and on the rates of shear strain between two orthogonal material line elements. The rates of rotation of line elements are also considered. However, the study of kinematics does not usually go beyond a consideration of the rate of deformation tensor, which determines the normal and shear strain rates, but does not measure all aspects of motion. Some higher order kinematic measures are introduced in this chapter.

Cartesian tensors have been used for developing the theory. In some instances a matrix representation for tensors has been used for convenience. A summary of the main results concerning Cartesian tensors, as well as a description of the notation, is given in the Appendix.

1.2 Velocity Gradient Tensor

The state of motion of a continuous body is assumed to be completely specified when the velocity distribution is known. Once the velocity \( v(x,t) \) of the particle at \( x \) is known at time \( t \), a study of the deformations of line elements may be attempted. Since deformations of material elements can be very large over finite time intervals, a study of deformations over infinitesimal time intervals is appropriate. This leads naturally to a study of the rates of deformation rather than deformations.

Consider a mass of fluid that has the configuration \( B \) at time \( t \) and \( B' \) at time \( t + \Delta t \). Let \( \mathbf{PQ} = d\mathbf{x} = d\xi \mathbf{n} \) be a material line element in \( B \) which, at time \( t \), is at the point \( x \), where \( \mathbf{n} \) is a unit vector along \( \mathbf{PQ} \), as shown in Fig. 1.2.1. After time \( \Delta t \), the same material line element, which will now be in configuration \( B' \), will move to \( \mathbf{P'Q'} = d\mathbf{X} = d\Xi \mathbf{N} \), where the point \( P' \) has the position vector \( \mathbf{X} \), and \( \mathbf{N} \) is a unit vector along \( \mathbf{P'Q'} \). The general problem of kinematics is then to relate the line element \( d\mathbf{X} \), at time \( t + \Delta t \),
1.2 Velocity Gradient Tensor

to its original state \( dx \), at time \( t \). Thus, the problem of studying short time deformations is to develop a relation of the form \( dX = f(dx, \Delta t) \). Let \( v \) and \( v + dv \) be the velocities of the points \( P \) and \( Q \), respectively, so that \( PP' = v \Delta t \) and \( QQ' = (v + dv) \Delta t \). Then, from the geometry shown in Fig. 1.2.1

\[
dX = dx + dv \Delta t
\]

(1.2.1)

![Diagram](image)

Fig. 1.2.1 Configurations of a body at two times.

Now \( dv = (dv_i) \) is the velocity difference \( v_Q - v_P \) of the points \( P \) and \( Q \) at the same time \( t \). Therefore, since at time \( t \) the velocity of the different fluid particles is a function of the coordinates \( x_i \) of \( x \),

\[
dv_i = \frac{\partial v_i}{\partial x_r} \, dx_r = v_{i,r} 
\]

Thus

\[
dv_i = g_{ri} dx_r \quad \text{or} \quad dv = G^T \, dx
\]

(1.2.2)

where \( G = (g_{ij}) = (v_{i,j}) \) is called the velocity gradient tensor.

Using this result, Eq. (1.2.1) gives

\[
dX_i = (\delta_{ni} + \Delta t \, g_{ni}) \, dx_r \quad \text{or} \quad dX = (I + \Delta t \, G^T) \, dx
\]

(1.2.3)