1. Introduction

The complex spacio-temporal development of turbulent flow has its origins in the nonlinear inertial couplings of the Navier-Stokes equations. At high Reynolds numbers these couplings lead to the continuous exchange of energy among a wide range of spatial scales and the creation of new scales. It is this process of scale generation and energy exchange which we discuss in this paper. Space necessarily limits the paper to primary results, and many interesting details must unfortunately await future publication.

Classically, the decomposition of the velocity field into well defined length scales has been accomplished through Fourier decomposition. A structure within the flow is identified as a band of Fourier modes in a narrow region of wave number space, its evolution described by the collective interactions with Fourier modes in all regions of Fourier space. Fourier decomposition of the Navier-Stokes equation leads directly to a triad structure in Fourier space upon which the nonlinear interactions are based. Thus, the triad interaction is the fundamental interaction in the nonlinear process of energy exchange, and this paper shall be devoted to triad evolution. The calculations bear directly on the issue of the correlation, or non-correlation between scales in different regions of the spectrum; and should find application in the interpretation of large scale numerical calculations using spectral methods [2,8].

The triad has been studied previously by Lee [3,4,5] in context with the development of ergodicity and mixing. In his analyses, however, energy is confined to the initial triad and new modes do not appear. Lorenz [7] simplified the dynamic equations to four specialized Fourier modes in two dimensions as a simple model of atmospheric zonal flow.

2. The Structure of the Navier-Stokes Equations in Fourier Space

2.1 Fourier Decomposition

In the Fourier spectral representation, the velocity field is represented as an infinite series of discrete complex Fourier modes with wave vector \( \mathbf{k} \) and complex amplitude \( \tilde{u}(k,t) \):

\[
\vec{v}(x,t) = \sum_{k} \tilde{u}(k,t) e^{i\mathbf{k} \cdot \mathbf{x}}
\]

(1)
Boundaries are sufficiently distant that the flow is considered unbounded. Equation (1) may therefore be interpreted as the discretized form of a Fourier integral, and periodicity will not be imposed. For incompressible flows, the complex amplitude is constrained by continuity to a plane perpendicular to its wave vector: \( \mathbf{k} \cdot \mathbf{u}(k,t) = 0 \). The condition that the velocity field be real requires that the wave vector set \([k]\) be separable into two sets \([+k]\) and \([-k]\) and that \( \hat{u}(-k) = \hat{u}^*(k) \).

The Fourier transform (1) of the Navier-Stokes equations, with continuity applied to eliminate pressure, leads to a set of coupled differential equations in time for the complex amplitudes \( \hat{u}(k,t) \) [vid.1,6]. These may be written

\[
\frac{d}{dt} + \nu k^2 \hat{u}(k,t) = -i \sum_{k'} \hat{u}(k') \cdot \mathbf{k} \cdot \mathbf{u}(k-k') \tag{2}
\]

The second term on the LHS is viscous dissipation, whereas the RHS describes the evolution of mode \( \hat{u} \) due to nonlinear interactions with all other modes \( \hat{u}' \).

\( \hat{u}(k') \cdot \mathbf{k} \) is the vectoral projection of the amplitude of mode \( \hat{u}' \) onto a plane perpendicular to \( \mathbf{k} \), and incorporates both the inertial exchange of energy between modes and the pressure term. The effect of pressure is to rotate the amplitude vector in a plane perpendicular to \( \mathbf{k} \), causing energy to be exchanged between components.

The physical structure of a complex Fourier mode represents what we call a 'complex shear wave', because it must be specified with two independent phase angles. This is in contrast to a 'simple shear wave' - a sine wave with single phase. Let us write \( \hat{u}(k) \) as a complex vector, \( \hat{u}(k) = \hat{a} + i\hat{b} \). Since continuity constrains \( \hat{a} \) and \( \hat{b} \) to a plane perpendicular to \( \mathbf{k} \), \( \hat{u}(k) \) is specified with 4 independent components. Splitting the wave vector set \([k]\) into its positive and negative parts and rewriting (1) over the positive set only, one may write the velocity field as a sum of complex shear waves:

\[
\hat{v}(x,t) = 2 \sum_{k} \left[ \hat{a}(k) \cos k \cdot x - \hat{b}(k) \sin k \cdot x \right] \tag{3}
\]

The two component sine waves produce a three dimensional velocity field described by a velocity vector which rotates around the wave vector in the spacial direction \( \mathbf{k} \). Changes in 'overall phase' of a complex shear wave arise both from changes in the relative magnitudes and the relative orientations of \( \hat{a} \) and \( \hat{b} \).

Like the velocity field, the vorticity field of a complex shear wave is in general three dimensional. However, when \( \hat{a} \) and \( \hat{b} \) are colinear, the two polar phase angles coincide and the complex shear wave reduces to a simple shear wave. It can be shown from (2), however, that if \( \hat{a} \) and \( \hat{b} \) are initially non-zero and colinear, they will not remain colinear--with the exception that when all wave vectors and amplitudes are in the same plane they remain confined to that plane. This leads to a purely two dimensional evolution with scalar vorticity.