7 Chiral Perturbation Theory (CHPT)

7.1 CHPT for the Meson Sector

Let us, for the sake of completeness, briefly repeat some facts which have been already stressed in this book in different contexts. The eight light hadrons \((\pi, K, \eta)\) are pseudoscalar mesons. They are believed to be composed of the quarks \((u, d, s)\) and their antiquarks. For yet unknown reasons, their masses \((m_u, m_d, m_s)\) happen to be small. If these masses were strictly zero, the Lagrangian of QCD would exhibit an exact \(SU(3)_R \times SU(3)_L\) symmetry. For the standard model to be consistent with the facts of life, it is crucial that the ground state of the theory is not symmetric under this group, such that this \(RL\) symmetry breaks down to \(SU(3)_{R+L}\). Pseudoscalar mesons are identified with the Goldstone bosons generated by the symmetry breakdown. If \(m_u, m_d\) and \(m_s\) were zero, we would have the chiral limit, and the pseudoscalar mesons would be massless. In real world, their masses are proportional to square roots of quark masses, e.g.

\[
M_\pi^2 = (m_u + m_d)B_0[1 + O(m)]
\]  

(7.1.1)

where the constant \(B_0\) depends on the quark condensate. The hidden symmetry reveals itself in the low energy properties of the pseudoscalar mesons. Weinberg (1979) showed that these properties can be analyzed on the basis of an effective Lagrangian, replacing the quark and gluon fields of QCD by a meson field, represented by an element in the \(SU(3)\) flavour gauge group. The effective Lagrangian is expanded in powers of derivatives of the meson field and in powers of mass matrix:

\[
m = \begin{pmatrix} m_d & m_d & m_s \end{pmatrix}
\]  

(7.1.2)

The leading contribution is of the form

\[
\mathcal{L}^{(2)} = \frac{F_0^2}{4} \langle \partial_\mu U^\dagger \partial_\mu U + 2B_0 m(U + U^\dagger) \rangle
\]  

(7.1.3)

where the symbol \(\langle A \rangle\) stands for the trace of the \(3\times3\) matrix \(A\). The constant \(F_0\) is the value of the pion decay constant \(F_\pi\) in the chiral limit

\[
F_0 = F_\pi [1 + O(m)]
\]  

(7.1.4)

The \(F_\pi\) is \(\simeq 93\) MeV. The various current algebra relations are readily obtained by
calculating the relevant tree graphs of $\mathcal{L}^{(2)}$. Write $U = \exp(i\varphi/F_0)$ and expand in powers of $\varphi$ which is a matrix given by $\varphi = \varphi^a \lambda^a/2$. An example is the well-known low energy theorem for the decay $\eta \rightarrow 3\pi$

$$A(s, t) = \frac{\sqrt{3} m_u - m_d}{4} \frac{s - \frac{4}{3} M_\pi^2}{F_0^2} + O(p^4) \quad (7.1.5)$$

where $m$ is the average mass of $u$ and $d$. The relation (5) is an exact statement in the following sense. The amplitude $A(s, t)$ is a function of the kinematic variables $s, t, \nu$ of the quark masses and the scale $\Lambda_{QCD}$. The electromagnetic contributions, of order $e^2 p^2$ are disregarded. Put a scale factor $\lambda$ in $s, t$ and the quark masses and expand in powers of $\lambda$. The theorem states that there is no term of order $\lambda^0$ and the contribution of order $\lambda^1$ is determined by the constants $F_0, B_0$ according to Eq. (5). Alternatively one may count the powers of four-momenta. Because of the Eq. (1) and the on mass shell constraints, such as $p^2 = M_\pi^2$, the quark masses must be counted as quantities of order $p^2$. The low energy theorem Eq. (5) determines the amplitude $A(s, t)$ at order $p^2$. The corrections are of order $p^4$.

More generally, the $T$-matrix elements describing the scattering of any number of pseudoscalar mesons are of order $p^2$. The leading contribution, given by the tree graphs of $\mathcal{L}^{(2)}$, only involve the quark masses and the two constants $F_0, B_0$. Unitarity implies that at order $p^4$, the $T$-matrix is not a polynomial in momenta, but involves cuts with discontinuities determined by the square of the leading contribution to the $T$-matrix. In fact together with analyticity, unitarity fixes the contribution of order $p^4$ upto a polynomial in the external momenta.

In the language of field theory, the discontinuities required by unitarity appear in the one-loop graphs of $\mathcal{L}^{(2)}$. Formally these graphs are represented by divergent integrals which require renormalization. The renormalized graphs are unambiguous only up to a polynomial in the external momenta. The occurrence of unspecified subtraction constants in the one-loop graphs reflects the fact that $\mathcal{L}^{(2)}$ only represents the leading term in the derivative expansion

$$\mathcal{L}_{\text{eff}} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \mathcal{L}^{(6)} + \quad (7.1.6)$$

of the full effective Lagrangian. The contribution $\mathcal{L}^{(4)}$ of order $p^4$ involves seven new coupling constants which are not fixed by $F_0, B_0$. If one restricts oneself to $T$-matrix elements to order $p^4$, there is no contribution from the higher terms $\mathcal{L}^{(6)} \ldots$. We will not go into details but will instead go to the external field method.

If the $T$-matrix elements are calculated beyond leading order it is not justified to identify $F_0$ with $F_\pi$. To exploit the experimental information concerning the matrix elements $\langle 0 | A_\mu | \pi \rangle \sim F_\pi$, it is useful to extend the frame work from a scheme which only deals with $T$-matrix elements to a scheme which allows one to calculate the Green's functions associated with the currents. The external

1 For a scattering process $AB \rightarrow CD$, we expect two independent kinetic variables. The Mandelstam variables in terms of four momenta are $s = (P_A + P_B)^2, t = (P_A - P_C)^2$ and $u = (P_A - P_D)^2$. The variables $s$ and $t$ correspond to processes $A\bar{D} \rightarrow C\bar{B}$ and $\bar{D}B \rightarrow C\bar{A}$. But $s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2$. 