3. Algebraic Approach to Classical Ergodic Theory

3.1 Abelian $C^*$ Dynamical Systems

Motivated by Section 2.3.1 we can think of a classical dynamical system in terms of an abstract Abelian $C^*$ algebra $\mathcal{A}$ (with identity $\hat{1}$) and of a reversible evolution defined by the $\mathbb{Z}$-action of a $^*$-automorphism $\Theta: \mathcal{A} \to \mathcal{A}$.

Topological considerations will involve the couple $(\mathcal{A}, \Theta)$, ergodic ones the dynamical triple $(\mathcal{A}, \Theta, \phi)$ where $\phi$ is any $\Theta$-invariant, normalized state, that is $\phi(\hat{1}) = 1, \phi \circ \Theta = \phi$.

In the following, quite a number of results will be quoted and used. For a full account of the underlying theory the reader is specially referred among the other possibilities cited in the text to [Ta], section I.3, and to [BR1], section 2.3.5.

Remark 3.1 The unit ball $S(\mathcal{A})$ of the dual space $\mathcal{A}^*$ of $\mathcal{A}$ and the subset $S_\Theta(\mathcal{A})$ of $\Theta$-invariant states are convex and compact with respect to the $w^*$-topology induced on $\mathcal{A}^*$ by $\mathcal{A}$. The subsets $\Xi(\mathcal{A})$, respectively $\Xi_\Theta(\mathcal{A})$, consisting of extremal, respectively extremal $\Theta$-invariant elements of $\mathcal{A}^*$, are non-void. The state space and its $\Theta$-invariant subset are then the $w^*$-closure of the linear convex hulls of their own extremal points. Each $\hat{a} \in \mathcal{A}$ can be identified with $G_a \in \mathcal{A}^{**}$ (the double dual), where the linear functional $G_a$ on $\mathcal{A}^*$ is defined by:

$$G_a(\phi) := \phi(\hat{a}) \quad (\phi \in \mathcal{A}^*).$$

The states $\phi$ of $\Xi(\mathcal{A})$ are extremal (pure), as such they are multiplicative linear functionals (characters), namely, $\phi(\hat{a}\hat{b}) = \phi(\hat{a})\phi(\hat{b})$, for $\hat{a}, \hat{b} \in \mathcal{A}$ (see [Th2, Par.2.2]). Then, the linear functionals $G_a \in \mathcal{A}^{**}$, when restricted to $\Xi(\mathcal{A})$, multiply together as functions:

$$(G_a G_b)(\phi) \equiv G_{ab}(\phi) = G_a(\phi) G_b(\phi) \quad (\hat{a}, \hat{b} \in \mathcal{A}, \phi \in \Xi(\mathcal{A})).$$

The map $G : \mathcal{A} \to \mathcal{A}^{**}$ is called Gelfand isomorphism and transforms each element of $\mathcal{A}$ into a function over $\Xi(\mathcal{A})$, continuous with respect to the $w^*$-topology. Therefore, any abstract Abelian $C^*$ algebra $\mathcal{A}$ with identity can
be thought of as a $C^*$ algebra of continuous functions over the compact space of its characters.

Furthermore, any automorphism $\Theta$ of $\mathcal{A}$ gives rise, by duality, to a homeomorphism $\Theta^*$ of $\mathcal{E}(\mathcal{A})$, $\Theta^*(\phi) = \phi \circ \Theta$, such that $G_{\Theta(a)}(\phi) = G_a(\Theta^*(\phi))$. There is thus no intrinsic difference between $(\mathcal{A}, \Theta)$ and the topological couple $(\mathcal{C}(\mathcal{X}), \Theta_T)$ of Section 2.3.1.

As to the ergodic triple, we need the particular point of view associated with an invariant state $\phi \in \mathcal{S}_\Theta(\mathcal{A})$. Riesz’s representation theorem tells us that expectation values of observables of $\mathcal{A}$ with respect to $\phi$ are given by integration with respect to an invariant probability measure $\mu_\phi$ defined on the Borel $\sigma$-algebra of $\mathcal{E}(\mathcal{A})$.

In Section 2.3.1 continuous functions are used to express ergodic and topological properties in terms of abstract (Abelian) $C^*$ algebras and of their states. We go one step further in abstractness by rephrasing those definitions as follows.

**Proposition 3.2** [NTW] The dynamical system $(\mathcal{A}, \Theta)$ is

1) Topologically transitive if and only if, for any two positive operators $\hat{a}, \hat{b} \in \mathcal{A}$, there exists $n \in \mathbb{N}$ such that $\hat{a} \Theta^n(\hat{b}) > 0$.
2) Topologically mixing if and only if there exists an $N \in \mathbb{N}$ such that $|n| > N$ implies $\hat{a} \Theta^n(\hat{b}) > 0$.

Once equipped with a $\Theta$-invariant state $\phi$, the dynamical system $(\mathcal{A}, \Theta)$ is

3) Ergodic if and only if $\phi$ belongs to $\mathcal{E}_\Theta(\mathcal{A})$ and is thus extremal invariant.
4) Mixing if and only if

$$\lim_{|n| \to \infty} \phi(\hat{a} \Theta^n(\hat{b})) = \phi(\hat{a})\phi(\hat{b}) \quad (\hat{a}, \hat{b} \in \mathcal{A}).$$

**Example 3.3** (One-Dimensional Ising Model) [NPT] The algebra of functions $\mathcal{U}_A$ of Example 2.32 is isomorphic to the algebra $\mathcal{D}_A := D_{2^n A}(\mathbb{C})$ of diagonal $2^n A \times 2^n A$ complex matrices. Like $\mathcal{U}_A$ into $\mathcal{U}$, $\mathcal{D}_A$ can be embedded into the infinite tensor product $\mathcal{D}_\infty := \bigotimes_{k \in \mathbb{Z}} (D_2(\mathbb{C}))_k$. The elements of $\mathcal{D}_\infty$ are tensor products of $2 \times 2$ diagonal matrices with all but a finite number of factors equal to the identity $\mathbb{I}_2$.

Let $V(A) := \{n_1 < n_2 < \ldots < n_A\}$ be an increasingly ordered $n$-tuple of sites and $\hat{a}_A := \bigotimes_{j=1}^A \hat{a}(n_j)$, $\hat{a}(n_j) \in D_2(\mathbb{C})$, an element of $\mathcal{D}_A$. We write it as an element of $\mathcal{D}_\infty$ by setting

$$\hat{a}_A := \hat{1}_{n_1 - 1} \otimes \hat{a}(n_1) \otimes \hat{1}_{[n_1 + 1, n_2 - 1]} \otimes \ldots \otimes \hat{a}(n_A) \otimes \hat{1}_{[n_A + 1}$$

$$\hat{1}_{[n, m]} := \bigotimes_{k=n}^m (\hat{1}_2)_k, \quad \hat{1}_{[-\infty, n]} := \hat{1}_{[n, +\infty]}.$$