Chapter 6

Forms of the Momentum Equation: Equations of Euler and Navier–Stokes

6-1 Main Differential Forms of the Momentum Equation

The momentum equation is obtained by equating the applied forces to the inertia force for a unit volume of the fluid. The physical meaning and the mathematical expressions of these forces have been developed in Chapters 4 and 5.

The different forms of the momentum equation corresponding to a number of cases encountered in hydrodynamics are now presented.

6-1.1 Perfect Fluid: Equations of Euler

6-1.1.1 The first major approximation is to assume that the fluid is perfect. In this case the friction forces are zero and the applied forces consist of gravity and pressure only. The momentum equation is obtained directly from the expressions developed in Chapters 4 and 5, in the three-axis system $OX, OY, OZ$, where $OZ$ is assumed to be vertical (see Table 6-1). When the expressions $du/dt$ and $p^*$ are

<table>
<thead>
<tr>
<th>Table 6-1</th>
<th>The momentum equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inertia forces ( \rho \frac{du}{dt} ) per unit of volume ( \rho \frac{dv}{dt} ) per unit of volume ( \rho \frac{dw}{dt} ) per unit of volume ( \frac{\partial p^<em>}{\partial x} ) per unit of volume ( \frac{\partial p^</em>}{\partial y} ) per unit of volume ( \frac{\partial p^*}{\partial z} ) per unit of volume</td>
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<tr>
<td>Pressure and gravity forces ( \rho \frac{dv}{dt} ) per unit of volume ( \rho \frac{dw}{dt} ) per unit of volume ( \frac{\partial p^<em>}{\partial x} ) per unit of volume ( \frac{\partial p^</em>}{\partial y} ) per unit of volume ( \frac{\partial p^*}{\partial z} ) per unit of volume</td>
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Written in vector notation, these become

\[
\rho \frac{dV}{dt} + \nabla p^* = 0
\]

† Recall \( p^* = p + \rho gz \).
Part 1: Establishing the Basic Equations that Govern Flow Motion

expanded (see Section 4-4.1), the momentum equation takes the form along the \( OX \) axis given by Equation 6-1.

\[
\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial}{\partial x}(p + \rho g z) \quad (6-1)
\]

Two similar equations may be written in the \( OY \) and \( OZ \) directions. These are called the equations of Euler.

Such a system of equations associated with the continuity relationship \( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \) forms the basis of the largest part of the hydrodynamics dealing with a perfect incompressible fluid. These equations are mathematically of the first order but are nonlinear (more specifically quadratic) because of the convective inertia terms. This quadratic term is the cause of most mathematical difficulties encountered in hydrodynamics.

6-1.1.2 It has been explained in Chapter 1 that it is possible to study hydrodynamic problems either in Eulerian coordinates or in Lagrangian coordinates. It is recalled that the Lagrangian method consists of following particles along their paths instead of dealing with particles at a given point. This method is used, for example, in some studies of periodic gravity waves over a horizontal bottom. The corresponding equations are given here only for the purpose of recognition in the literature and will not be developed.

If \( X, Y, Z \) are the volume or body forces, i.e., gravity, the Lagrangian equation along the \( OX \) axis is written:

\[
\frac{1}{\rho} \frac{\partial p}{\partial x_0} = \left( X - \frac{\partial^2 x}{\partial t^2} \right) \frac{\partial x}{\partial x_0} + \left( Y - \frac{\partial^2 y}{\partial t^2} \right) \frac{\partial y}{\partial y_0} + \left( Z - \frac{\partial^2 z}{\partial t^2} \right) \frac{\partial z}{\partial z_0}
\]

Two similar equations give the value of \( \frac{\partial p}{\partial y_0} \) and \( \frac{\partial p}{\partial z_0} \) by permutation of \( x_0, y_0, z_0 \), which are the coordinates of a particle at time \( t = t_0 \). These are called the equations of Lagrange.

6-1.2 Viscous Fluid and the Navier–Stokes Equations

6-1.2.1 If the friction forces are introduced in the Eulerian equations, the Navier–Stokes equations are obtained (see Section 5-4.1), as shown in Equation 6-2.

The Navier–Stokes equations are the basis of most problems in fluid mechanics dealing with liquid. They are second-order differential equations because of the friction terms, and nonlinear because of the convective inertia terms.

6-1.2.2 These Navier–Stokes equations are written in a very concise manner with the aid of tensorial notation. Although a knowledge of tensorial calculus is not required, it is given here as a guide to further reading on the subject.

Use is made of two subscripts, \( i \) and \( j \), which indicate when an operation is to be systematically repeated and which component of a vector quantity (such as \( V \)) is being considered. When an index is repeated in a term, the considered quantity has to be summed over the possible components. For example, the continuity equation \( \frac{\partial u_i}{\partial x_i} + \frac{\partial v_j}{\partial y_j} + \frac{\partial w_i}{\partial z_i} = 0 \) is tensorially written: \( \frac{\partial u_i}{\partial x_i} = 0 \), since the subscript \( i \) indicates that the quantity (here \( V \)) has to be summed over the three components \( OX, OY, OZ \).

The three previous Navier–Stokes equations, may be written simply as:

\[
\rho \left[ \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right] = -\frac{\partial}{\partial x_i} (p + \rho g z) + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j}
\]

Here, the subscript \( i \) is called “free index” and indicates the component being considered; the subscript \( j \), called “dummy index,” indicates repeated operations.