5.1 $L^2(\mathbb{R})$ and Approximate Identities

Despite the previous few chapters, the term "wavelets" usually refers to wavelets on $\mathbb{R}$, examples of which we construct in this chapter. The first two sections present the basics of Fourier analysis on $\mathbb{R}$.

We consider complex-valued functions defined on $\mathbb{R}$. As one might suspect from chapter 4, to obtain a suitable notion of orthogonality we must restrict ourselves to functions are not too large. Specifically, we consider functions $f$ that are square-integrable, that is, such that

$$\int_{\mathbb{R}} |f(x)|^2 \, dx < +\infty.$$ 

As in the case of $L^2(-\pi, \pi)$ in chapter 4, we are using the Lebesgue integral and identifying two functions that agree a.e. (almost everywhere). (Also as in chapter 4, the reader unfamiliar with these terms can just ignore them, if he or she is willing to accept a few consequences of this theory.) Formally,

$$L^2(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{C} : \int_{\mathbb{R}} |f(x)|^2 \, dx < +\infty \right\}.$$
$L^2(\mathbb{R})$ is a vector space with the operations of pointwise addition and scalar multiplication of functions (Exercise 5.1.1 (i)).

For $f, g \in L^2(\mathbb{R})$, define

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x)g(x) \, dx.$$  \hfill (5.1)

By Exercise 5.1.1(ii), $\langle \cdot, \cdot \rangle$ is an inner product on $L^2(\mathbb{R})$. Applying Definition 1.90 and Exercise 1.6.5, $L^2(\mathbb{R})$ is a normed space with the norm

$$\|f\| = \left( \int_{\mathbb{R}} |f(x)|^2 \, dx \right)^{1/2},$$  \hfill (5.2)

called the $L^2$ norm. The Cauchy-Schwarz inequality (Lemma 1.91) gives us

$$\left| \int_{\mathbb{R}} f(x)g(x) \, dx \right| \leq \left( \int_{\mathbb{R}} |f(x)|^2 \, dx \right)^{1/2} \left( \int_{\mathbb{R}} |g(x)|^2 \, dx \right)^{1/2},$$

for $f, g \in L^2(\mathbb{R})$. By applying this inequality with $f$ and $g$ replaced by $|f|$ and $|g|$, respectively, we obtain

$$\int_{\mathbb{R}} |f(x)g(x)| \, dx \leq \left( \int_{\mathbb{R}} |f(x)|^2 \, dx \right)^{1/2} \left( \int_{\mathbb{R}} |g(x)|^2 \, dx \right)^{1/2}. \hfill (5.3)$$

Also, by Corollary 1.92, we have the triangle inequality

$$\left( \int_{\mathbb{R}} |f(x) + g(x)|^2 \, dx \right)^{1/2} \leq \left( \int_{\mathbb{R}} |f(x)|^2 \, dx \right)^{1/2} + \left( \int_{\mathbb{R}} |g(x)|^2 \, dx \right)^{1/2}. \hfill (5.4)$$

We define convergence in $L^2(\mathbb{R})$ in accordance with the definitions in section 4.2 for a general inner product space. Namely, suppose $\{f_n\}_{n \in \mathbb{N}}$ is a sequence of functions in $L^2(\mathbb{R})$ and $f \in L^2(\mathbb{R})$. We say $\{f_n\}_{n \in \mathbb{N}}$ converges to $f$ in $L^2(\mathbb{R})$ if, for all $\epsilon > 0$, there exist $N \in \mathbb{N}$ such that $\|f_n - f\| < \epsilon$ for all $n > N$. By Exercise 4.2.1, this is equivalent to $\|f_n - f\| \to 0$ as $n \to +\infty$. We say $\{f_n\}_{n \in \mathbb{N}}$ is Cauchy if, for all $\epsilon > 0$, there exists $N$ such that $\|f_n - f_m\| < \epsilon$ for all $n, m > N$. We assume the somewhat deep fact (which depends on properties of the Lebesgue integral) that $L^2(\mathbb{R})$ is complete, meaning that every Cauchy sequence in $L^2(\mathbb{R})$ converges in $L^2(\mathbb{R})$. Thus, in the terminology of section 4.2, $L^2(\mathbb{R})$ is a Hilbert space. In particular,