II. Solution of the Dirichlet Problem for a Schlicht Region

§ 1. The Poisson Integral

1.1. As an essential tool in the following discussion we shall use some results from the theory of harmonic functions. The basis of these theorems is our ability to solve the first boundary value problem, i.e., our ability to construct a harmonic function with preassigned boundary values in a region $G$. In this section we shall apply the results of the first chapter to establish the following special case of this general theorem.

Suppose a real value $u(\zeta)$ is prescribed at every boundary point $z = \zeta$ of a schlicht region $G$ of connectivity $p$ in the $z$-plane. Furthermore, assume that $G$ is bounded by the Jordan curves $\Gamma_1, \ldots, \Gamma_p$ and that the function $u(\zeta)$ is bounded on each arc $\Gamma_r$ and, except for at most a finite number of points, is continuous. Then there exists one and only one bounded, harmonic function on the interior of $G$ whose values tend to the boundary value $u(\zeta)$ as $z$ tends to a point of continuity $z = \zeta$.

1.2. If $G$ is the unit disk $|z| < 1$, then, as was first shown by H. A. Schwarz [1] the boundary value problem is solved by the Poisson integral

$$ u(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\phi}) \frac{1 - r^2}{1 + r^2 - 2r \cos (\theta - \phi)} d\phi. \quad (1.1) $$

We saw in I, § 1, that the kernel of the integral is a harmonic function of the point $z = re^{i\theta}$, and the same is therefore true for the Poisson integral at every point of the unit disk. To see that the function defined by the integral tends to the prescribed boundary value $u(\zeta)$ as $z \to \zeta$, it is convenient to first consider the more special boundary value problem, where $u(\zeta)$ assumes the value 1 on an arc $\overrightarrow{\zeta_1 \zeta_2}$ ($\zeta_1 = e^{i\theta_1}$, $\zeta_2 = e^{i\theta_2}$) of the unit circle of length $\theta_2 - \theta_1 = \alpha$ $(0 < \alpha \leq 2\pi)$ and vanishes on the complementary arc. The Poisson integral is then

$$ \frac{1}{2\pi} \int_{\theta_1}^{\theta_2} \frac{1 - r^2}{1 + r^2 - 2r \cos (\theta - \phi)} d\theta, $$

but this is the "harmonic measure" $\omega(z; \theta_1, \theta_2)$ of the arc $\langle \theta_1, \theta_2 \rangle$, which we defined on p. 7 and which has the required boundary properties, as was shown there.
After this preliminary remark we return to the general integral (1.1), which can be written in the form

\[ u(z) = \int_{\theta=0}^{2\pi} u(e^{i\theta}) \, d\omega(z;0,\theta). \]  

(1.1')

Suppose that \( \zeta = \zeta_0 = e^{i\theta_0} \) is a point of continuity of \( u(\zeta) \), that \( \varepsilon \) is an arbitrarily small positive number and that \( \theta_1 \leq \theta \leq \theta_2 \) is an interval containing the value \( \theta_0 \) where the variation of \( u(\zeta) \) is smaller than \( \varepsilon \). We then have

\[
u(z) - u(\zeta_0) = \int_{\theta=0}^{2\pi} (u(e^{i\theta}) - u(\zeta_0)) \, d\omega \\
= \int_{\theta_1}^{\theta_2} \langle \varepsilon \rangle \, d\omega + \int_{\theta_1}^{\theta_2+2\pi} (u(e^{i\theta}) - u(\zeta_0)) \, d\omega,
\]

where \( \langle \varepsilon \rangle \) stands for a quantity whose magnitude is \( < \varepsilon \). Here the absolute value of the first term on the right is less than \( \varepsilon \), while the magnitude of the second term is at most

\[ 2M \omega(z; \theta_2, \theta_1 + 2\pi), \]

where \( M \) stands for the maximum of \( |u(\zeta)| \). Now as \( z \) tends to the point \( \zeta = e^{i\alpha} \), where \( \theta_1 < \theta_0 < \theta_2 \), the harmonic measure \( \omega(\zeta; \theta_2, \theta_1 + 2\pi) \) vanishes uniformly. Consequently, it is possible to find a number \( q_\varepsilon \) so that if \( |z - \zeta_0| < q_\varepsilon \), then \( \omega < \varepsilon \); for these same values of \( z \),

\[ |u(z) - u(\zeta_0)| < \varepsilon (2M + 1), \]

which shows that \( u(z) \) tends uniformly to the prescribed boundary value \( u(\zeta_0) \) as \( z \rightarrow \zeta_0 \).

At every point of discontinuity \( \zeta_0 \) where the left and right hand limits \( u(e^{i(\theta_0+\delta)}) \) and \( u(e^{i(\theta_0-\delta)}) \) exist, \( u(z) \) has the limit

\[ \lambda u(e^{i(\theta_0+\delta)}) + (1 - \lambda) u(e^{i(\theta_0-\delta)}) \]

when \( z \) approaches \( \zeta_0 \) along a path which makes an angle \( \lambda \tau \) with the positive tangent to the circle \( |z| = 1 \). The proof is analogous to the above proof for a point of continuity.

1.3. It remains to be shown that the Poisson integral \( u(z) \) is the only bounded solution for \( |z| < 1 \) to the boundary value problem we have posed. Let \( u_1(z) \) be any given solution; the difference \( v(z) = u(z) - u_1(z) \) then defines a bounded harmonic function for \( |z| < 1 \), and it vanishes