QUALITATIVE BEHAVIOR OF THE SOLUTIONS OF
PERIODIC FIRST ORDER SCALAR DIFFERENTIAL EQUATIONS
WITH STRICTLY CONVEX COERCIVE NONLINEARITY

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1. Introduction.

It has been proved in [4] that if \( f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is continuous, \( f(\cdot, u) \) is \( T \)-periodic for each \( u \in \mathbb{R} \), \( f(x, \cdot) \) is strictly convex on \( \mathbb{R} \) for each \( x \in \mathbb{R} \), and if \( f(x, \cdot) \) is uniformly coercive, i.e.
\[
f(x, u) \to +\infty \text{ as } |u| \to \infty
\]
uniformly in \( x \in \mathbb{R} \), then there exists \( s_1 \in \mathbb{R} \) such that the equation
\[
(1) \quad u'(x) + f(x, u(x)) = s
\]
has exactly zero, one or two \( T \)-periodic solutions according to \( s < s_1 \), \( s = s_1 \) or \( s > s_1 \). The aim of this note is to complete the result by getting a fairly complete picture of the trajectoires of (1) under the same assumptions upon \( f \). Our results apply in particular to the forced Bernoulli equation with periodic coefficients
\[
(2) \quad u'(x) + a_1(x)u(x) + a_2(x)u^{2k}(x) = a_0(x)
\]
where the \( a_i : \mathbb{R} \to \mathbb{R} \) are continuous and \( T \)-periodic, \( a_0(x) > 0 \) and \( k \) is a positive integer (and to its special case of the Riccati equation) and describes accurately the qualitative behavior of their solutions according to the values of
\[
s = \frac{1}{(1/T)} \int_0^T a_0(x)dx.
\]
Applications can be made to the equations of deterministic models for the growth of populations subject to periodic fluctuations and periodic harvesting and the reader can consult [2] and [5] for some specific contributions to this problem.
2. The structure of the set of solutions when $s > s_1$.

Let us consider the periodic differential equation

$$(2_s) \quad u'(x) + f(x, u(x)) = s$$

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, $f(, u)$ is $T$-periodic for each $u \in \mathbb{R}$, $f(x, .)$ is strictly convex for each $x \in \mathbb{R}$ and $f(x, u) \to +\infty$ as $|u| \to \infty$

uniformly in $\mathbb{R}$. Let $s_1 \in \mathbb{R}$ be the real such that $(2_s)$ has exactly zero, one or two $T$-periodic solutions according to $s < s_1$, $s = s_1$ or $s > s_1$ (see [4] for the proof of this result). We shall assume in this section that $s > s_1$ and denote the $T$-periodic solutions of $(2_s)$ by $u_s$ and $v_s$ respectively. Since the assumptions above imply that the Cauchy problem for $(2_s)$ is locally uniquely solvable (see the proof in [4]), we can assume, without loss of generality that

$$u_s(x) < v_s(x)$$

for all $x \in \mathbb{R}$. It will be convenient to associate to $f$ the function

$$R_f : \mathbb{R} \times (\mathbb{R}^2 \setminus \Delta) \to \mathbb{R},$$

$$(x, u, v) \to \frac{f(x, u) - f(x, v)}{u - v},$$

where $\Delta = \{(u, v) \in \mathbb{R}^2 : u = v\}$ is the diagonal in $\mathbb{R}^2$. The strict convexity assumption made upon $f$ is equivalent to the fact that for each $x$, $u$ or $v$ fixed, the functions $R_f(x, ., v)$ and $R_f(x, u, .)$ are increasing on $\mathbb{R}$. We shall use the following simple observation.

**Lemma 1.** If $(2)$ satisfies the conditions listed above and if $s > s_1$, then, for each $x_0 \in \mathbb{R}$ we have

$$\int_{x_0}^{x_0 + T} R_f(x, v_s(x), u_s(x))dx = 0,$$

where $u_s$ and $v_s$ denote the two $T$-periodic solutions of $(2_s)$.