2. Some Examples of Linear Groups

In this chapter we shall have a look at what sort of group has a faithful representation of finite degree over a field. Our plan of campaign is roughly as follows. Firstly we consider abelian groups, and then soluble groups—especially those satisfying either the minimal condition or the maximal condition on subgroups. Next we prove a local theorem of Mal’cev, which states that a group has a faithful representation of degree \( n \) if and only if each of its finitely generated subgroups has such a representation. Then we prove that a free group has a faithful representation of degree 2 over most fields, and discuss similar results for certain relatively free groups. We follow this with Nisnevich’s Theorem, that a free product of groups having faithful representations of degree \( n \) over fields of characteristic \( p \geq 0 \), has a faithful representation of degree \( n + 1 \) over some field of characteristic \( p \), plus some related results. Finally we consider the representability of wreath products of linear groups.

We begin with a trivial remark.

2.1. If \( G \) is any finite group and \( F \) is any field, then \( G \) has a faithful representation over \( F \) of finite degree. ⊥

If \( G \) is any group \( \tau(G) \) is the subgroup of \( G \) generated by all the periodic normal subgroups of \( G \); that is \( \tau(G) \) is the maximum periodic normal subgroup of \( G \). \( G \) has finite rank at most \( n \) if every finite subset of \( G \) is contained in an \( n \)-generator subgroup of \( G \). If \( G \) is abelian and periodic then \( G \) has finite rank at most \( n \) if and only if for each prime \( p \) the Sylow \( p \)-subgroup of \( G \) is a direct product of at most \( n \) cyclic and Prüfer \( p \)-groups (a Prüfer \( p \)-group is a \( \mathbb{C}_{p^\infty} \)-group). If \( \pi \) is any set of primes and \( G \) is a group with a unique maximal \( \pi \)-subgroup we denote this maximal \( \pi \)-subgroup by \( G_\pi \).

2.2. Theorem (Mal’cev [34]). i) An abelian group \( A \) has a faithful representation of degree \( n \geq 1 \) over some field of characteristic zero if and only if \( \tau(A) \) has rank at most \( n \).

ii) An abelian group \( A \) has a faithful representation of degree \( n \geq 1 \) over some field of characteristic \( p > 0 \) if and only if \( \tau(A)_p \) has finite rank \( r \) and
\(\tau(A)_p\) has finite exponent \(p^e\) satisfying
\[p^{e-1} + \max \{1, r\} < n + 1.\]

**Proof:** i) Let \(A\) be an abelian subgroup of \(GL(n, F)\) where \(\text{char} F = 0\). We can clearly suppose that \(F\) is algebraically closed. By 1.6 and 1.3, \(\tau(A)\) is diagonalizable; that is, there exists an element \(x\) of \(GL(n, F)\) such that \(\tau(A)x \subseteq D(n, F)\). Now \(D(n, F)\) is isomorphic to the direct product of \(n\) copies of \(F^*\) and the Sylow subgroups of \(F^*\) are Prüfer groups. Therefore \(\tau(A)\) has finite rank at most \(n\).

Suppose now that \(A\) is an abelian group such that \(\tau(A)\) has rank at most \(n\). Let \(F\) be an algebraically closed field of characteristic zero of transcendence degree greater than \(\max \{\aleph_0, |A|\}\). Denote by \(\mathcal{S}\) the set of all pairs \((B, \phi)\) where \(B\) is a subgroup of \(A\) and \(\phi\) is a monomorphism of \(B\) into \(D(n, F)\). \(\mathcal{S}\) is non-empty, \((\{1\}, 1 \mapsto 1) \in \mathcal{S}\). Order \(\mathcal{S}\) by defining \((B_1, \phi_1) \preceq (B_2, \phi_2)\) whenever \(B_1 \subseteq B_2\) and \(\phi_2|_{B_1} = \phi_1\). \(\mathcal{S}\) is inductively ordered. Let \((C, \psi)\) be a maximal element of \(\mathcal{S}\); this exists by Zorn’s Lemma. If \(C \neq A\) there exists an element \(a\) of \(A \setminus C\) such that either

1. \(aC\) has infinite order in \(A/C\) or
2. \(aC\) has order \(q\) in \(A/C\) for some prime \(q\).

Suppose that \(|aC| = \infty\). Then \(\langle a, C \rangle = \langle a \rangle \times C\).

\[C \psi \subseteq D(n, F_1) \subseteq D(n, F)\]

for some subfield \(F_1\) of \(F\) of cardinal \(\max \{\aleph_0, |A|\}\). Hence there exists an element \(\alpha\) of \(F\) that is algebraically independent of \(F_1\). Define

\[\psi_1: \langle a, C \rangle \to D(n, F) \quad \text{by} \quad (a^t c) \psi_1 = \alpha^t(c \psi).\]

\(\psi_1\) is a monomorphism of \(\langle a, C \rangle\) into \(D(n, F)\) such that \(\psi = \psi_1|_C\). This contradiction of the maximality of \((C, \psi)\) proves that \(C = A\).

2. Suppose that \(a^q = b \in C\) where \(q\) is a prime. If \(a_1\) is an element of \(A\) satisfying \(a_1^q = b\), then \((a_1 a^{-1})^q = 1\). Since the rank of \(\tau(A)\) is at most \(n\), \(A\) contains at most \(q^n\) elements \(a_1\) satisfying \(a_1^q = b\), one of which is \(a\). Hence \(C\) contains at most \((q^n - 1)\) elements \(a_1\) satisfying \(a_1^q = b\). \(D(n, F)\) contains \(q^n\) elements \(x_1\) satisfying \(x_1^q = b \psi\) since \(F\) is algebraically closed, and \(C \psi\) can contain at most \(q^n - 1\) of these \(x_1\). Therefore there exists \(x\) in \(D \setminus C \psi\) satisfying \(x^q = b\). It is simple to check that the mapping

\[\psi_1: \langle a, C \rangle \to D(n, F)\]

given by \((a^t c) \psi_1 = x^t(c \psi)\) is a monomorphism extending \(\psi\). As in Case 1 this contradiction proves that \(A = C\) and completes the proof of part i) of the theorem.

ii) Let \(A\) be an abelian subgroup of \(GL(n, F)\) where \(F\) is an algebraically closed field of characteristic \(p > 0\). If the element \(a\) of \(A\) has order \(p^e\) then