IV. A Predicate Calculus

§ 1. Preliminary remarks about the rules of the predicate calculus. The concept of derivability

1.1 The relation \( \alpha_1, \ldots, \alpha_n \vdash \alpha \), or \( \vdash \alpha_1 \ldots \alpha_n \). As we have already explained in Chap. I, § 5, we want to lay down a calculus with the help of which we can obtain algorithmically all the consequences of arbitrary sets of expressions. Various different calculi of this sort are known today; every such calculus is called a predicate calculus or, more precisely, a first-order predicate calculus. (For predicate calculus of higher order cf. Chap. VI, § 1.) In the next section, we shall give a particularly simple calculus of this sort. For the sake of simplicity, we shall call this calculus the predicate calculus (instead of a predicate calculus).

The predicate calculus we shall introduce here takes the form of an assumption calculus (for this, cf. the remarks in Chap. I, § 5.5). The rows of symbols which we can derive consist of linear sequences of finitely many expressions, where every such sequence must consist of at least one expression. Thus, every row of symbols which is derivable in the predicate calculus has the form

\[ \alpha_1 \ldots \alpha_r \]

with \( r \geq 1 \). The connection with the notion of consequence, which was our reason for creating the calculus, can be described by two assertions:

1. If \( \alpha_1 \ldots \alpha_r \) is derivable in the predicate calculus, then \( \alpha_r \) follows from \( \alpha_1, \ldots, \alpha_{r-1} \) (in the particular case that the sequence consists of a single element \( \alpha_1 \), this is to mean that \( \alpha_1 \) follows from the empty set, i.e. that \( \alpha_1 \) is universally valid).

2. If, conversely, an expression \( \alpha \) follows from a set \( \mathcal{W} \), then there are finitely many elements \( \alpha_1, \ldots, \alpha_s \) of \( \mathcal{W} \) such that \( \alpha_1 \ldots \alpha_s \alpha \) is derivable in the predicate calculus.
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From this, we see that the last expression of a derivable sequence of expressions has a different status from the preceding ones. We want to call attention to this by using

\[ (**) \alpha_r \text{ is derivable from } \alpha_1, \ldots, \alpha_{r-1} \text{ in the predicate calculus as a synonym for} \]

\[ (*) \alpha_1 \ldots \alpha_r \text{ is derivable in the predicate calculus.} \]

(If the sequence \( \alpha_1, \ldots, \alpha_r \) consists of only one element, then the two sentences are the same.) In analogy to the symbol "\( \vdash \)" with which we represent the relation of consequence, we shall represent the relation of derivability by "\( \vdash \)".

"\( \vdash \)" was introduced by Frege, the most important logician of the last century (1848-1925), in his "Begriffsschrift" (1879) to denote the relation of inference.

Thus, we abbreviate (\( * \)) and (\( ** \)) by:

\[ (***) \vdash \alpha_1 \ldots \alpha_r \text{ and } \alpha_1, \ldots, \alpha_{r-1} \vdash \alpha_r \text{ respectively} \]

(for \( r = 1 \), we have \( \vdash \alpha_1 \) in both cases).

In order to make the special status of the last element of a sequence of expressions obvious, we shall sometimes write \( \alpha_1 \ldots \alpha_{r-1} \vdash \alpha_r \) with a small gap between the last two expressions, instead of \( \alpha_1 \ldots \alpha_{r-1} \vdash \alpha_r \). We shall also often write \( \alpha_1 \ldots \alpha_{r-1} \vdash \alpha_r \). (This colon is, strictly speaking, not a component of the row of symbols; it is there merely for the convenience of the reader.)

1.2 The relation \( \mathfrak{M} \vdash \alpha \). In order to make the analogy between the relation of consequence and that of derivability even clearer, we shall introduce the syntactic relation \( \mathfrak{M} \vdash \alpha \) (in words: \( \alpha \) is derivable from the set of expressions \( \mathfrak{M} \)) in analogy to the semantic relation \( \mathfrak{M} \models \alpha \) by means of the

Definition. \( \mathfrak{M} \vdash \alpha \text{ if and only if there are finitely many elements } \alpha_1, \ldots, \alpha_s \ (s \geq 0) \text{ such that } \alpha_1, \ldots, \alpha_s \vdash \alpha \).

The main result of theoretical predicate logic can now be formulated as follows:

\[ (3) \mathfrak{M} \vdash \alpha \text{ if and only if } \mathfrak{M} \models \alpha. \]

It states that the semantic relation of consequence and the syntactic relation of derivability coincide with each other.