CHAPTER III.3

Infinitely Many $\delta'$-Interactions in One Dimension

Now we derive the main results of Ch. 2 for $\delta'$ instead of $\delta$-interactions. We shall closely follow the strategy for $\delta$-interactions and only present detailed proofs if the arguments differ substantially from those in Ch. 2.

Let $J \subset \mathbb{Z}$ be the index set of Sect. 2.1 and $Y = \{y_j \in \mathbb{R} | j \in J\}$ be a discrete subset of $\mathbb{R}$ satisfying (2.1.1) and the remarks after (2.1.1). In analogy to Ch. II.3 we introduce the minimal operator $\hat{H}_Y$ in $L^2(\mathbb{R})$

$$\hat{H}_Y = -\frac{d^2}{dx^2}, \quad \mathcal{D}(\hat{H}_Y) = \{g \in H^{2,2}(\mathbb{R}) | g'(y_j) = 0, y_j \in Y, j \in J\}. \quad (3.1)$$

Then $\hat{H}_Y$ is closed and nonnegative and its adjoint operator reads

$$\hat{H}_Y^* = -\frac{d^2}{dx^2}, \quad \mathcal{D}(\hat{H}_Y^*) = \{g \in H^{2,2}(\mathbb{R} - Y) | g'(y_j -) = g'(y_j +), y_j \in Y, j \in J\}. \quad (3.2)$$

The equation

$$\hat{H}_Y^* \phi(k) = k^2 \phi(k), \quad \phi(k) \in \mathcal{D}(\hat{H}_Y^*), \quad k^2 \in \mathbb{C} - \mathbb{R}, \quad \text{Im } k > 0, \quad (3.3)$$

then has the solutions

$$\phi_j(k, x) = \begin{cases} e^{ik(x-y_j)}, & x > y_j, \\ -e^{ik(y_j-x)}, & x < y_j \end{cases}, \quad \text{Im } k > 0, \quad y_j \in Y, \quad j \in J, \quad (3.4)$$

which span the deficiency subspace of $\hat{H}_Y$. Thus $\hat{H}_Y$ has deficiency indices $(\infty, \infty)$. According to Appendix C a particular type of self-adjoint extensions...
of $H_Y$ is of the type

$$\Xi_{\beta, Y} = -\frac{d^2}{dx^2}, \quad \mathcal{D}(\Xi_{\beta, Y}) = \{ g \in H^{2,2}(\mathbb{R} - Y) | g'(y_j+) = g'(y_j-), g(y_j+) - g(y_j-) = \beta g'(y_j), j \in J \},$$

$$\beta = \{ \beta_j \}_{j \in J}, -\infty < \beta_j \leq \infty, \quad j \in J. \quad (3.5)$$

By definition $\Xi_{\beta, Y}$ describes $\delta'$-interactions of strength $\beta_j$ centered at $y_j \in Y, j \in J.$ The special case $\beta_j = 0, j \in J,$ leads to the kinetic energy operator $-\Delta$ on $H^{2,2}(\mathbb{R})$ whereas the case $\beta_{j_0} = \infty$ for some $j_0 \in J$ leads to a Neumann boundary condition at the point $y_{j_0}$ (i.e., $g'(y_{j_0}+) = g'(y_{j_0}-) = 0$).

Since Theorems 2.1.1 and 2.1.2 immediately go through with $-\Delta_{x, Y}, -\Delta_{x_{M,N}, Y_{M,N}}$ replaced by their respective analogs, we directly proceed to a description of the resolvent of $\Xi_{\beta, Y}.$

**Theorem 3.1.** Let $\beta_j \in \mathbb{R} - \{0\}, j \in J,$ and assume (2.1.1). Then

$$(\Xi_{\beta, Y} - k^2)^{-1} = G_k + \sum_{j,j' \in J} [\mathcal{G}_{\beta, Y}(k)]_{jj'}^{-1}(\mathcal{G}_k(\cdot - y_j), \cdot) \mathcal{G}_k(\cdot - y_{j'}),$$

$$k^2 \in \rho(\Xi_{\beta, Y}), \quad \text{Im} \ k > 0. \quad (3.6)$$

Here

$$\mathcal{G}_{\beta, Y}(k) = [-((\beta_j k^2)^{-1} \delta_{jj'} + G_k(y_j - y_{j'}))]_{j,j' \in J}, \quad \text{Im} \ k > 0, \quad (3.7)$$

is a closed operator in $l^2(Y)$ with

$$[\mathcal{G}_{\beta, Y}(k)]^{-1} \in \mathcal{B}(l^2(Y)), \quad k^2 \in \rho(\Xi_{\beta, Y}), \quad \text{Im} \ k > 0 \quad \text{large enough}, \quad (3.8)$$

and

$$\mathcal{G}_k(x - y) = (i/2k) \begin{cases} e^{ik(x-y)}, & x > y, \\ -e^{ik(y-x)}, & x < y, \end{cases} \quad (3.9)$$

**Proof.** One can follow the proof of Theorem 2.1.3 step by step since obviously $|\mathcal{G}_k(x)| = |G_k(x)|.$

The analog of Theorem 2.1.4 then reads

**Theorem 3.2.** Let $\beta_j \in \mathbb{R} - \{0\}, j \in J,$ and assume (2.1.1). Then the domain $\mathcal{D}(\Xi_{\beta, Y})$ consists of all elements $\psi$ of the type

$$\psi(x) = \phi_k(x) + (i/k) \sum_{j,j' \in J} [\mathcal{G}_{\beta, Y}(k)]^{-1}_{jj'} \phi_k'(y_j) \mathcal{G}_k(x - y_j), \quad (3.10)$$

where $\phi_k \in \mathcal{D}(-\Delta) = H^{2,2}(\mathbb{R})$ and $k^2 \in \rho(\Xi_{\beta, Y}), \quad \text{Im} \ k > 0.$ The decomposition (3.10) is unique and with $\psi \in \mathcal{D}(\Xi_{\beta, Y})$ of this form we obtain

$$(\Xi_{\beta, Y} - k^2)\psi = (-\Delta - k^2)\phi_k. \quad (3.11)$$

Next let $\psi \in \mathcal{D}(\Xi_{\beta, Y})$ and suppose that $\psi = 0$ in an open set $U \subseteq \mathbb{R}.$ Then $\Xi_{\beta, Y}\psi = 0$ in $U.$