solutions for which the asymptotic expansion has been given in section 4.3. It is seen that, among others, the infinitesimal perturbations belong to the class of admissible initial values. Moreover, the conclusion remains valid if in the course of time new perturbations are introduced, that is if for example in Eqs. (4.6.7) we change the constants $C_n$ an arbitrary number of times and in an arbitrary fashion, provided that at all times the order of magnitude requirement (4.6.2), (4.6.3) is satisfied.

Chapter 5

ANALYSIS OF SOME ONE-DIMENSIONAL PROBLEMS

5.1 Introductory remarks

In the preceding chapters we have considered a rather wide class of non-linear partial differential equations and in developing methods of analysis for the non-linear stability problems we have made various assumptions, concerning the outcome of the linearized theory, the behaviour of coefficients, etc. We shall presently illustrate these developments by some specific examples, which show that the hypothesis which have been made are realistic, that is, that in certain problems of interest the assumptions are indeed verified.

On the other hand, since our results are concerned mostly with asymptotic expansions, a comparison with some exact solutions would be of fundamental interest. Unfortunately, this author was unable to construct a mathematical problem belonging to the class that has been defined, for which he could obtain an exact solution. We shall however compare the results, in one of the two problems treated below, to approximate solutions obtained by an entirely different method.

In present chapter we study first J. M. Burgers' famous mathematical model of turbulence. This permits us to illustrate the developments of sections 4.3 and 4.5. In the final section we consider another simple mathematical model, which falls in the class of section 4.4.

5.2 Burgers' mathematical model of turbulence

In a series of papers, summarized in Burgers (1948), the following problem was proposed and studied:

Let $\Phi$ be a function of $\eta$ and $t$, which satisfies the equation

$$\frac{\partial \Phi}{\partial t} - \Phi - \frac{1}{R} \frac{\partial^2 \Phi}{\partial \eta^2} = - \frac{\partial \Phi^2}{\partial \eta} + U \Phi \quad (5.2.1)$$

W. Eckhaus, Studies in Non-Linear Stability Theory
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where $R$ is a constant, while $U$ is a function of $t$, defined by

$$
\frac{dU}{dt} + \frac{1}{R} U = -\int_0^1 \Phi^2 d\eta \tag{5.2.2}
$$

The boundary conditions require that $\Phi = 0$ for $\eta = 0$ and $\eta = 1$.

In his analysis Burgers has demonstrated that the behaviour of $\Phi$ exhibits striking analogies with the behaviour of perturbations in hydrodynamic turbulence. In what follows we concentrate on the stability aspects of Burgers' model. First we summarize some of Burgers' results which will be needed in our analysis.

Let us seek stationary solutions of Eqs. (5.2.1), (5.2.2) subject to the given boundary conditions. Burgers has shown that for large values of the parameter $R$ asymptotic solutions can be obtained; in the first approximation for $R \to \infty$ Burgers finds

$$
\Phi \approx \frac{1}{2} (1 + U) \{\eta - \tanh[\frac{1}{2} R (1 + U) \eta]\} \tag{5.2.3}
$$

$$
\Phi \approx \frac{1}{2} (1 + U) \{(\eta - 1) - \tanh[\frac{1}{2} R (1 + U) (\eta - 1)]\} \tag{5.2.4}
$$

These solutions are of 'boundary-layer type', and can further be analysed by decomposition in Fourier series.

We write:

$$
\Phi(\eta) = \sum_{n=0}^{\infty} A_n \sin(n + 1) \pi \eta \tag{5.2.5}
$$

Corresponding to solution (5.2.3) there is the spectrum

$$
A_n \approx \frac{\pi}{R \sinh \left[ \frac{\pi^2 (n + 1)}{R (1 + U)} \right]} \tag{5.2.6}
$$

To the solution (5.2.4) corresponds the spectrum

$$
A_n \approx \frac{(-1)^{n+1} \pi}{R \sinh \left[ \frac{\pi^2 (n + 1)}{R (1 + U)} \right]} \tag{5.2.7}
$$

Now Eqs. (5.2.3) to (5.2.6) still contain the quantity $U$, which has to be calculated from the formula

$$
U = -R \int_0^1 \Phi^2 d\eta \tag{5.2.8}
$$

It can be shown that for $R \to \infty$ one obtains

$$
U \approx -1 + 2 \sqrt{\frac{3}{R}} \tag{5.2.9}
$$