§ 15
Normality of \( m \)-completions

Let \( B \) a ring and \( A \) a subring of \( B \). An element \( x \in B \) is said to be integral over \( A \) if there are \( a_0, \ldots, a_{n-1} \in A \) such that \( a_0 + \cdots + a_{n-1} x^{n-1} + x^n = 0 \) \((n > 0)\). The ring \( A \) is said to be integrally closed in \( B \) if every element of \( B \) which is integral over \( A \) is an element of \( A \). Finally a domain \( A \) is said to be integrally closed if \( A \) is integrally closed in its quotient field.

**Definition 15.1.** A ring \( A \) is said to be normal if and only if \( A_p \) is an integrally closed domain for any \( p \in \text{spec} \ A \).

**Examples.** a) A domain is normal if and only if it is integrally closed (see e.g. [3], p. 63, prop. 5.13).

b) A noetherian ring is normal if and only if it is a direct product of integrally closed domains. This follows easily from the definition and Lemma 10.3.

c) A factorial ring is integrally closed (see e.g. [3], p. 63). It follows that any locally factorial ring is normal. In particular any regular ring is normal (Th. 11.9).

Now we recall Serre's normality criterion for noetherian rings.

**Theorem 15.2.** A noetherian ring is normal if and only if it is \( R_1 \) and \( S_2 \).

**Proof.** See, e.g. [22], p. 108, Th. 5.8.6. \( \Box \)

By Serre's criterion and Theorems 13.5 and 13.11 we get immediately the following result.

**Corollary 15.4.** Let \( \varphi: A \to B \) be a flat homomorphism of noetherian rings. Then:

(i) If \( B \) is normal and \( \varphi \) is faithfully flat, \( A \) is normal.

(ii) If \( A \) and the fibers of \( \varphi \) are normal, then \( B \) is normal.
Now we return to \( m \)-adic completions. We say that a local ring \( A \) is \textit{analytically normal} if its completion \( \hat{A} \) is normal. If \( A \) is \textit{noetherian}, then \( A \) itself must be normal by Theorem 4.9 and Corollary 15.4. (i).

By the usual techniques we get the following theorems.

\textbf{Theorem 15.5.} Let \( A \) be a noetherian \( m \)-ring and \( \hat{A} \) its completion. Then:

(i) If \( A_n \) is analytically normal for any maximal ideal \( n \) containing \( m \), then \( \hat{A} \) is normal.

(ii) If moreover \( \text{spec}(A/m) \) is connected, then \( \hat{A} \) is a normal domain.

\textit{Proof.} The first part follows from Proposition 6.5, by means of the usual techniques. The second follows easily by Proposition 10.8 and Example a).

\textbf{Theorem 15.6.} Let \( A \) be a noetherian \( m \)-ring, and \( \hat{A} \) its completion. Suppose further that the formal fibers of \( A \) are normal for any maximal ideal \( n \) containing \( m \). Then the following conditions are equivalent:

(i) \( \hat{A} \) is normal;

(ii) \( A_n \) is normal for any maximal ideal \( n \) containing \( m \);

If moreover \( m \subset \text{rad} A \), then the above conditions are equivalent to:

(iii) \( A \) is normal.

\textit{Proof.} It follows by 13.5, 13.12 and 15.2.

\textit{Remark.} Since a regular ring is normal, the above theorem can be applied to all rings described in Corollary 12.8, since they have geometrically regular formal fibers.

Now we want to apply the above results to the rings of restricted power series. We need a well known lemma.

\textbf{Lemma 15.7.} Let \( A \) be a ring. Then the polynomial ring \( A[X_1, \ldots, X_n] \) is normal if and only if \( A \) is normal.

\textit{Proof.} It is an easy consequence of [7], p. 20, Cor. 3.