3. FORMULATION OF MATHEMATICAL PROBLEMS

With a specific reaction-diffusion system at hand, one generally wishes either to find a solution satisfying certain subsidiary conditions, or to determine certain qualitative properties shared by many solutions. The various possibilities here generate a variety of mathematical problems, the most important of which are described in this chapter.

The broadest framework within which we shall operate is the following generalization of (1.20) to more than one space dimension (to conform more with convention, we shift notation from $p$ to $u$):

$$\frac{\partial u_i}{\partial t} = L_i u_i + F_i(x,t,u), \quad i = 1, \ldots, n,$$

(3.1)

where the $L_i$ are uniformly elliptic differential operators in the variables $x = (x_1, \ldots, x_n)$, of the form

$$L_i u \equiv -\sum \frac{\partial}{\partial x_k} H_{ik} - H_{ik} = \sum D_{ikl}(x,t) \frac{\partial u}{\partial x_l} + C_{ik}(x,t)u.$$

3.1 The standard problems

Let $\Omega$ be a domain in $m$-space (we often take $m = 1$, in which case $\Omega$ is an interval). When $m > 1$ and when $\Omega$ is not the whole space, we assume its boundary $\partial \Omega$ to have a unit normal which is a smooth function of the position on $\partial \Omega$.

For $T > 0$, let $Q_T = \Omega \times (0,T)$.

Initial boundary value problems: Find a continuous function $u: \overline{Q_T} \to \mathbb{R}^n$, bounded in $x$ for each fixed $t \in (0,T)$, the derivatives appearing in (3.1) existing and continuous in $\Omega \times (0,T]$, and satisfying

(i) (3.1) in $Q_T$,

(ii) initial condition $u(x,0) = \phi(x)$, where $\phi$ is a prescribed vector function, bounded and continuous on $\partial \Omega$,

(iii) boundary conditions on $\partial \Omega \times (0,T]$, of a type described in Section

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1.3 for the case \( m = 1 \). Specifically, we have

\[
    u_1(x,t) = c_1(x,t) \quad \text{(Dirichlet condition)} \tag{3.2a}
\]

\[
    \sum \sum_{k} v_k(x)H_{1k} = a_1(x,t) \quad \text{(Neumann condition)} \tag{3.2b}
\]

\[
    \sum \sum_{k} v_k(x)H_{1k} = a_1(x,t) + b_1(x,t)u \quad \text{(Robin condition).} \tag{3.2c}
\]

Here \( \gamma(x) = (\nu_1, \ldots, \nu_m) \) is the unit outward normal at \( x \), and the \( b_1 > 0 \).

**Initial value problem:** The same, with \( \Omega = \mathbb{R}^m \), and no boundary conditions.

**Boundary value problems:** Find a bounded function \( u: \Omega \times (-\infty,0) \rightarrow \mathbb{R}^n \), continuous together with the derivatives appearing in (3.1), satisfying (3.1) in \( \Omega \times (-\infty,0) \), and one of the three boundary conditions on \( \partial \Omega \times (-\infty,0) \).

**Stationary problems:** Find a time-independent solution of the boundary-value problem, together with one of the boundary conditions (3.2); the prescribed functions and coefficients are now supposed to be time-independent.

**Periodic problems:** Supposing the functions in (3.1) and (3.2) are periodic in time with common period, find a solution of a boundary value problem which is periodic in time with the same period.

In most of what we shall do, the question of stability will be an important, though usually not an easy, question. Most of the existing results pertain to \( C^0 \)-stability. Accordingly, we define

\[
    |u(\cdot,t)|_0 \equiv \sup_{x \in \Omega \setminus \Omega_i} |u_i(x,t)|.
\]

**Definition:** Let \( u(x,t) \) be a solution of (3.1) on \( \Omega \times (0,\infty) \), satisfying a boundary condition of a type (3.2). It is \( C^0 \)-stable if, given \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that every solution \( u^* \) of (3.1) on \( \Omega \times (0,T) \) for some \( T > 0 \) satisfying the same boundary condition and satisfying \( |u^*(\cdot,0) - u(\cdot,0)|_0 < \delta \), (i) can be continued to be a solution of (3.1), with the same boundary condition, on \( \Omega \times (0,\infty) \), and (ii) \( |u^*(\cdot,t) - u(\cdot,t)|_0 < \varepsilon \) for all \( t > 0 \).