7. Some Basic Properties of Matter Waves

7.1 Wave Packets

In the two preceding chapters it was shown that light, electrons and other elementary particles can have both wave and particle characteristics. In this chapter we will examine more closely how the wave properties of matter can be understood and described mathematically.

For both light and material particles there are basic relationships between energy and frequency, and between momentum and wavelength, which are summarised in the following formulae:

\begin{align}
\text{Light} & \quad E = h \nu \\
& \quad p = \frac{h \nu}{c} \\
\text{Matter} & \quad E = h \nu = h \omega \\
& \quad p = \frac{h}{\lambda} = h k .
\end{align}

(7.1)

Fig. 7.1. Instantaneous view of a wave with amplitude $A_0$ and wavelength $\lambda$

We now wish to expand these relationships into a more exact theory. We are familiar with descriptions of wave motion from the study of light. If we consider a plane monochromatic wave (Fig. 7.1) travelling in the $x$ direction, the wave amplitude $A$ at time $t$ and point $x$ is $A(x, t) = A_0 \cos(kx - \omega t)$. The wave number $k$ is related to the wavelength $\lambda$ by $k = 2 \pi / \lambda$. The circular frequency $\omega$ is related to the frequency by $\omega = 2 \pi \nu$. In many cases it is more useful to use complex notation, in which we express the cosine by exponential functions according to the formula

$$\cos \alpha = \frac{1}{2}(e^{i \alpha} + e^{-i \alpha}).$$

(7.2)
We accordingly expand $A(x, t)$:

$$A(x, t) = A_0 \frac{1}{2} \left[ \exp(ikx - i \omega t) + \exp(-ikx + i \omega t) \right]. \quad (7.3)$$

Applying the relations (7.1), we obtain

$$\exp(ikx - i \omega t) = \exp \left[ \frac{i}{\hbar} (px - Et) \right]. \quad (7.4)$$

The wave represented by (7.4) is an infinitely long wave train.

On the other hand, since we ordinarily assume that particles ("point masses") are localised, we must consider whether we can, by superposing a sufficient number of suitable wave trains, arrive at some spatially concentrated sort of "wave". We are tempted to form what are called wave packets, in which the amplitude is localised in a certain region of space. In order to get an idea of how such wave packets can be built up, we first imagine that two wave trains of slightly differing frequencies and wave-numbers are superposed. We then obtain from the two amplitudes $A_1(x, t)$ and $A_2(x, t)$ a new amplitude $A(x, t)$ according to

$$A(x, t) = A_1(x, t) + A_2(x, t), \quad (7.5)$$

or, using cosine waves of the same amplitude for $A_1$ and $A_2$,

$$A(x, t) = A_0 [\cos(k_1x - \omega_1 t) + \cos(k_2x - \omega_2 t)]. \quad (7.6)$$

As we know from elementary mathematics, the right-hand side of (7.6) may be expressed as

$$2A_0 \cos(kx - \omega t) \cos(\Delta kx - \Delta \omega t), \quad (7.7)$$

where

$$k = \frac{1}{2}(k_1 + k_2), \quad \omega = \frac{1}{2}(\omega_1 + \omega_2),$$

and

$$\Delta k = \frac{1}{2}(k_1 - k_2), \quad \Delta \omega = \frac{1}{2}(\omega_1 - \omega_2).$$

The resulting wave is sketched in Fig. 7.2. The wave is clearly amplified in some regions of space and attenuated in others. This suggests that we might produce a more and more complete localisation by superposing more and more cosine waves. This is, in fact, the case. To see how, we use the complex representation. We superpose waves of the form (7.4) for various wavenumbers $k$ and assume that the wavenumbers form a continuous distribution. Thus, we form the integral

$$\int_{k_0 - \Delta k}^{k_0 + \Delta k} a \exp[i(kx - \omega t)] \, dk = \psi(x, t), \quad (7.8)$$

where $a$ is taken to be a constant amplitude.